"JUST THE MATHS"

SLIDES NUMBER

14.11

PARTIAL DIFFERENTIATION 11 (Constrained maxima and minima)

by

A.J.Hobson

UNIT 14.11
PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

We consider the determination of local maxima and local minima for a function, $f(x, y, ...)$, subject to an additional constraint in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique for elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$f(x, y) \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

Here, it is possible to eliminate either x or y by using the constraint.

If we eliminate x, we may write $f(x, y)$ as a function, $F(y)$, of y only.

$f(x, y) \equiv F(y) \equiv 3(1 - 2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$

Using the principles of maxima and minima for functions of a single independent variable,

$$F'(y) \equiv 28y - 12 \quad \text{and} \quad F'' \equiv 28.$$

A local minimum occurs when $y = 3/7$ and, hence, $x = 1/7$.

The corresponding local minimum value of $f(x, y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x, y, z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x, y, z)$ as a function, $F(y, z)$, of y and z only.

$$f(x, y, z) \equiv F(y, z) \equiv (1 - 2y - 3z)^2 + y^2 + z^2.$$

That is,

$$F(y,z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \;\text{ and }\; \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z.$$

A stationary value will occur when these are both equal to zero.
Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$.
The corresponding value of x is $1/14$, which gives a stationary value for $f(x,y,z)$ of $14/(14)^2 = \frac{1}{14}$.
Also,

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \;\; \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \;\text{ and }\; \frac{\partial^2 F}{\partial y \partial z} \equiv 12.$$

Thus,

$$\frac{\partial^2 F}{\partial y^2} . \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2 + y^2 + z^2$, subject to the constraint that $x + 2y + 3z = 1$, at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad \text{and} \quad z = \frac{3}{14}.$$

Note:
Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x + 2y + 3z = 1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x, y, ...)$ subject to the constraint that $g(x, y, ...) = 0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x, y, ...$

The following example illustrates an alternative method for a function of two independent variables:

(a) Suppose that the function $z \equiv f(x, y)$ is subject to the constraint that $g(x, y) = 0$.

Then, z is effectively a function of x only.

The stationary values of z will be determined by the equation,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0.$$

(b) The total derivative of $z \equiv f(x, y)$ with respect to x, when x and y are not independent of each other, is given by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From $g(x, y) = 0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence, for all points on the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right) \frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will occur when

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x} = 0.$$

This may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then, $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \quad \text{and} \quad g(x, y) = 0.$$

Conclusion

The stationary values of the function $z \equiv f(x, y)$, subject to the constraint that $g(x, y) = 0$, occur at the points for which the function,

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y),$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**

Notes:

(i) To determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function, $z \equiv 3x^2 + 2y^2$, subject to the constraint that $x + 2y - 1 = 0$.

   Solution

   Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

   Then,

$$\frac{\partial \phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 4y + 2\lambda \text{ and } \frac{\partial \phi}{\partial \lambda} \equiv x + 2y - 1.$$

   The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

   Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$.

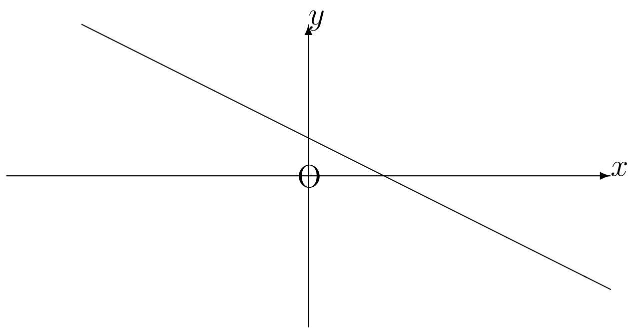
   Substituting this into the constraint, $7x - 1 = 0$.

$$\text{Hence, } x = \frac{1}{7}, \quad y = \frac{3}{7} \text{ and } \lambda = -\frac{6}{7}.$$

A single stationary point occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

The geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line with equation, $x + 2y - 1 = 0$.

For example, at the points $0.12, 0.44)$ and $(0.16, 0.42)$ on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

Solution

Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial\phi}{\partial x} \equiv 3+2\lambda x, \quad \frac{\partial\phi}{\partial y} \equiv 4+2\lambda y \quad \text{and} \quad \frac{\partial\phi}{\partial\lambda} \equiv x^2+y^2-1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \quad \text{and} \quad y = \frac{2}{\lambda}.$$

Substituting into the constraint gives

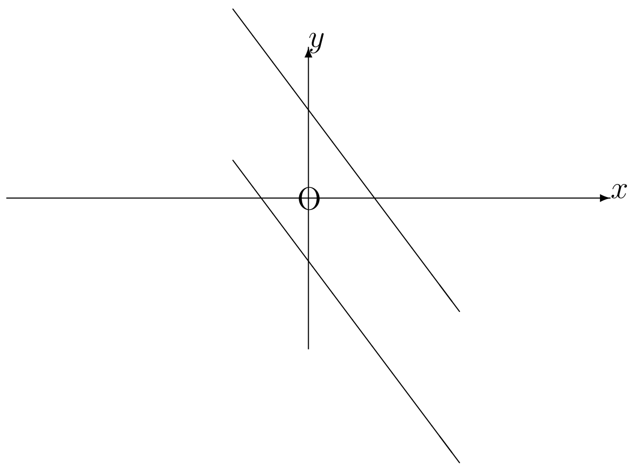
$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \quad \text{and} \quad \text{hence} \quad \lambda = \pm\frac{5}{2}.$$

Hence, $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.

Finally, the geometrical conditions suggest that we consider a straight line with equation,
$3x+4y=c$, (a constant) moving across the circle with equation, $x^2+y^2=1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$, will occur where the straight line touches the circle.

We have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2+y^2+z^2,$$

subject to the constraint that $x+2y+3z=1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda$$

and

$$\frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$.

Substituting these into the constraint gives $14x = 1$.

Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \text{ and } \lambda = -\frac{1}{7}.$$

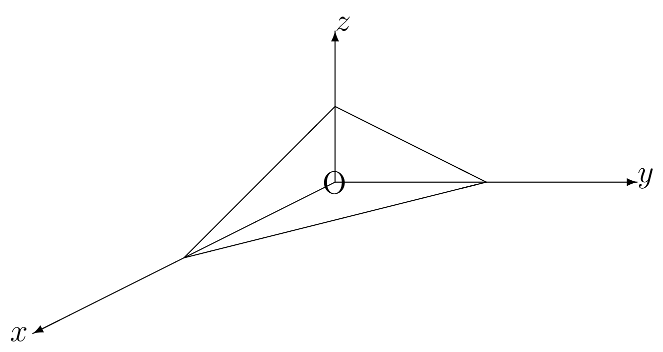
A single stationary point occurs, therefore, at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14}$$

and

$$w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2+y^2+z^2$, represents the square of the distance of a point (x, y, z) from the origin.

If the point is constrained to lie on a plane, this distance is bound to have a minimum value.