"JUST THE MATHS"

SLIDES NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

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UNIT 14.10
PARTIAL DIFFERENTIATION 10

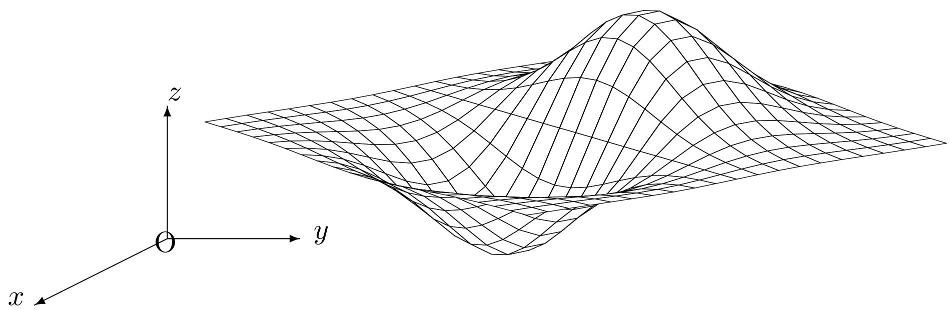
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

The equation,

$$z = f(x, y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x, y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x, y)$, is said to have a **"local maximum"** at a point, $\text{P}(x_0, y_0, z_0)$, if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

DEFINITION 3

The function, $z = f(x, y)$, is said to have a **"local minimum"** at a point, $\text{P}(x_0, y_0, z_0)$, if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x, y)$, in the neighbourhood of P.

Note:

At a stationary point, $\text{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

The following conditions are stated without proof:

(a) Sufficient conditions for a local maximum

A point, $\mathrm{P}(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, P(x_0, y_0, z_0), on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are.

(iii) The values of z at the local maxima and local minima of the function $z = f(x, y)$ may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \text{ and } \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, - - - - - - - - - - - - (1) \\ -x + 2y &= 0. - - - - - - - - - - - - (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \quad \text{or} \quad 3x(2x+1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0, \quad z = 4$ and $y = -\frac{1}{4}, \quad z = -\frac{65}{16}$, respectively. The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 3 > 0$$

and, therefore, the point $(0, 0, 4)$ is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \ \frac{\partial^2 z}{\partial y^2} > 0, \ \text{and} \ \frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -3 < 0$$

and, therefore, the point $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$ is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \ \text{ and } \ \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, we solve the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - - (2) \end{aligned}$$

From equation (2), $y = 0$ or $4x = 3y$.

Putting $y = 0$ in equation (1) gives $x = \pm 5$.

Stationary points occur at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into (1) gives $y = \pm 4$, $x = \pm 3$.
Stationary points occur at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points,

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y.$$

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-pt.
$(-3, -4, 300)$	-36	36	-48	negative	saddle-pt.

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion.

Outline Proof of the Sufficient Conditions

From Taylor's theorem for two variables,

$$f(a+h,b+k)-f(a,b)=hf_x(a,b)+kf_y(a,b)+$$

$$\frac{1}{2}\left(h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right)+\ \ldots,$$

where h and k are small compared with a and b.

f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a,b)=0$ and $f_y(a,b)=0$, the conditions for a local minimum at the point $(a,b,f(a,b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form,

$$Lh^2 + 2Mhk + Nk^2,$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If $LN - M^2 < 0$, it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a, b)$, $f_{yy}(a, b)$ and $f_{xy}(a, b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.