"JUST THE MATHS"

SLIDES NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

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UNIT 14.1
PARTIAL DIFFERENTIATION 1

PARTIAL DERIVATIVES
OF THE FIRST ORDER

14.1.1 FUNCTIONS OF
SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1.

$$z = f(x, y).$$

2.

$$w = F(x, y, z).$$

Normally, the variables on the right-hand side are called the **"independent variables"**.

The variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

ILLUSTRATION

$$x^2 + y^2 + z^2 = 16.$$

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, may not always be independent of one another

ILLUSTRATION

In the formula

$$z = xy^2 + \sin(x - y),$$

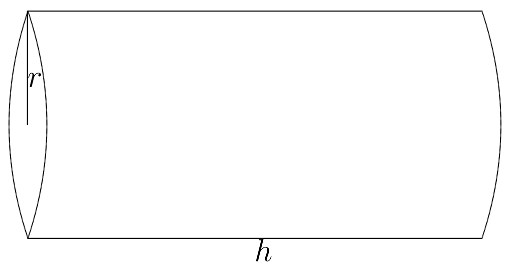
suppose that $x = t - 1$ and $y = 3t + 2$;

Then the variables, x and y, are not independent of each other.

In fact, $y = 3(x + 1) + 2 = 3x + 5.$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION



The volume, V, and the surface area, S, of a solid right-circular cylinder with radius r and height h are given by

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h.$$

V and S are functions of r and h.

Suppose r is held constant while h is allowed to vary.

Then,

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r \text{ const.}} = 2\pi r.$$

These are the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose h is held constant while r is allowed to vary.

Then,

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h \text{ const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h \text{ const.}} = 4\pi r + 2\pi h.$$

These are the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

This is indicated by

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

   (a)

$$z = (x^2 + 3y)^5.$$

Solution

$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4.2x = 10x(x^2 + 3y)^4$$

and

$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4.3 = 15(x^2 + 3y)^4.$$

   (b)

$$w = ze^{3x-7y}.$$

Solution

$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

and

$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

(c)

$$z = x \sin(2x^2 + 5y).$$

Solution

$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2 \cos(2x^2 + 5y)$$

and

$$\frac{\partial z}{\partial y} = 5x \cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. If

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

(where z is the dependent variable), determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

$$-\sin(x + 2z).\left(1 + 2\frac{\partial z}{\partial x}\right) + 2y\left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x\left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$