"JUST THE MATHS"

SLIDES NUMBER

10.7

DIFFERENTIATION 7
(Inverse hyperbolic functions)

by

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UNIT 10.7 - DIFFERENTIATION 7

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

10.7.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric and inverse hyperbolic functions should be considered as standard results, as follows:

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}},$$
where $-\infty < \sinh^{-1}x < \infty$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}},$$
where $\cosh^{-1}x \geq 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2}$$
where $-\infty < \tanh^{-1}x < \infty$.

10.7.2 THE DERIVATIVE OF AN INVERSE HYPERBOLIC SINE

We shall consider the formula

$$y = \text{Sinh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case S in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to

$$x = \sinh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y \equiv \sqrt{1 + \sin^2 y} \equiv \sqrt{1 + x^2},$$

noting that $\cosh y$ is never negative.

Thus,

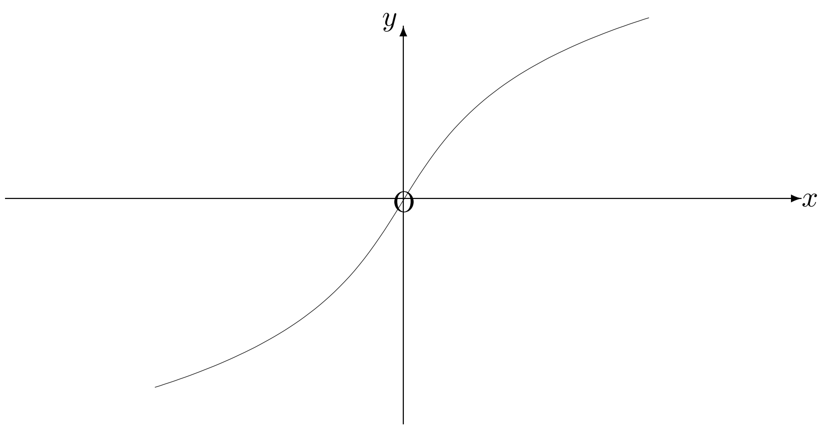
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 + x^2}}.$$

Consider now the graph of the formula

$$y = \text{Sinh}^{-1}x,$$

which may be obtained from the graph of $y = \sinh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x may lie anywhere in the interval $-\infty < x < \infty$.
2. For each value of x, the variable y has only one value.
3. For each value of x, there is only one possible value of $\frac{dy}{dx}$, which is positive.

4. There is no need to distinguish between a general value and a principal value of the inverse hyperbolic sine function since there is only one value of both the function and its derivative.

   However, it is customary to denote the inverse function by $\sinh^{-1}x$ using a lower-case s.

   Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}}.$$

10.7.3 THE DERIVATIVE OF AN INVERSE HYPERBOLIC COSINE

We shall consider the formula

$$y = \mathrm{Cosh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; see later.

The formula is equivalent to

$$x = \cosh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y \equiv \pm\sqrt{\cosh^2 y - 1} \equiv \pm\sqrt{x^2 - 1}.$$

Thus,

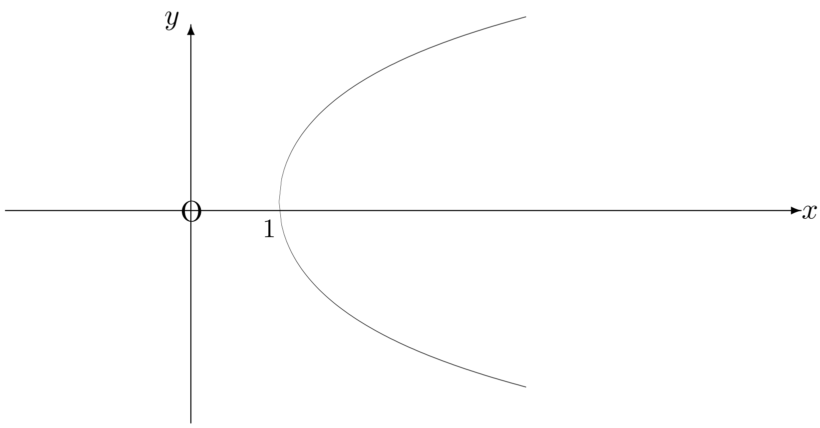
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cosh}^{-1}x,$$

which may be obtained from the graph of $y = \cosh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x must lie in the interval $x \geq 1$.
2. For each value of x in the interval $x > 1$, the variable y has two values one of which is positive and the other negative.

3. For each value of x in the interval $x > 1$, there are only two possible values of $\frac{dy}{dx}$, one of which is postive and the other negative.

4. On the part of the graph for which $y \geq 0$, there will be only one value of y with one (positive) value of $\frac{dy}{dx}$ for each value of x in the interval $x \geq 1$.

   The restricted part of the graph defines the **"principal value"** of the inverse cosine function and is denoted by $\cosh^{-1}x$ using a lower-case c.

   Hence,

$$\frac{d}{dx}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2 - 1}}.$$

10.7.4 THE DERIVATIVE OF AN INVERSE HYPERBOLIC TANGENT

We shall consider the formula

$$y = \text{Tanh}^{-1}x$$

and determine an expression for $\frac{dy}{dx}$.

Note:
The use of the upper-case T in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to

$$x = \tanh y;$$

so,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{sech}^2 y \equiv 1 - \tanh^2 y \equiv 1 - x^2.$$

Thus,

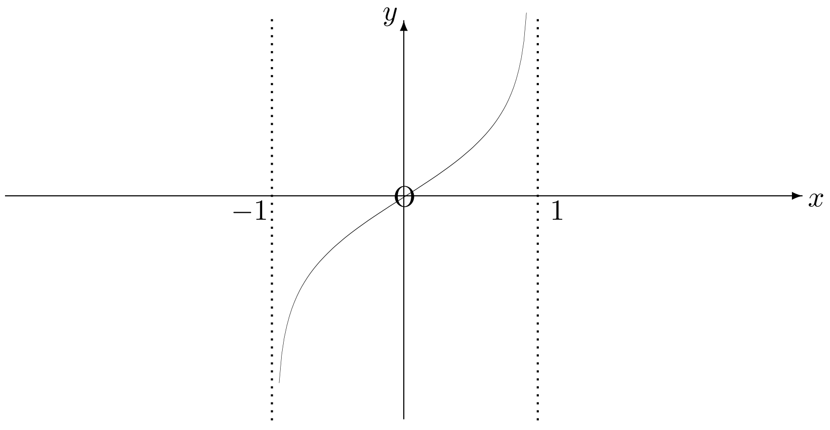
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tanh}^{-1} x,$$

which may be obtained from the graph of $y = \tanh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

We obtain



Observations

1. The variable x must lie in the interval $-1 < x < 1$.
2. For each value of x in the interval $-1 < x < 1$, the variable y has just one value.
3. For each value of x in the interval $-1 < x < 1$, there is only possible value of $\frac{dy}{dx}$, which is positive.
4. As with $\sinh^{-1}x$, there is no need to distinguish between a general value and a principal value of the inverse hyperbolic tangent; but it is customary to denote it by $\tan^{-1}x$ (lower-case t).

$$\text{Hence } \frac{d}{dx}[\tanh^{-1}x] = \frac{1}{1-x^2}.$$

ILLUSTRATIONS

1.

$$\frac{d}{dx}[\sin^{-1}(\tanh x)] = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \text{sech} x.$$

2.

$$\frac{d}{dx}[\cosh^{-1}(5x-4)] = \frac{5}{\sqrt{(5x-4)^2-1}},$$

assuming that $5x - 4 \geq 1$; that is, $x \geq 1$.