"JUST THE MATHS"

SLIDES NUMBER

10.6

DIFFERENTIATION 6
(Inverse trigonometric functions)

by

A.J.Hobson

UNIT 10.6 - DIFFERENTIATION 6

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

10.6.1 SUMMARY OF RESULTS

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}},$$

where $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}},$$

where $0 \leq \cos^{-1}x \leq \pi$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2},$$

where $-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$.

10.6.2 THE DERIVATIVE OF AN INVERSE SINE

We shall consider the formula

$$y = \mathrm{Sin}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case S in the formula (see later).

The formula is equivalent to

$$x = \sin y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \equiv \pm\sqrt{1 - \sin^2 y} \equiv \pm\sqrt{1 - x^2}.$$

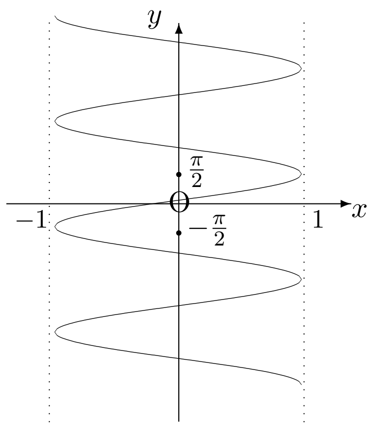
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1 - x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sin}^{-1}x.$$

This may be obtained from the graph of $y = \sin x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

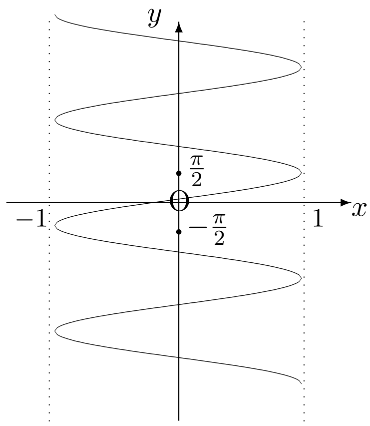


Observations

(i) x must lie in the interval $-1 \leq x \leq 1$.

(ii) For each x in $-1 \leq x \leq 1$, y has infinitely many values - spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one positive and the other negative.



(iv) On the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y and one (positive) value of $\frac{dy}{dx}$ for each x in $-1 \leq x \leq 1$.

The restricted part of the graph defines the **"principal value"** of the inverse sine function and is denoted by $\sin^{-1}x$ using a lower-case s.

Hence,

$$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

10.6.3 THE DERIVATIVE OF AN INVERSE COSINE

We shall consider the formula

$$y = \mathrm{Cos}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula (see later).

The formula is equivalent to

$$x = \cos y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \equiv \pm\sqrt{1-\cos^2 y} \equiv \pm\sqrt{1-x^2}.$$

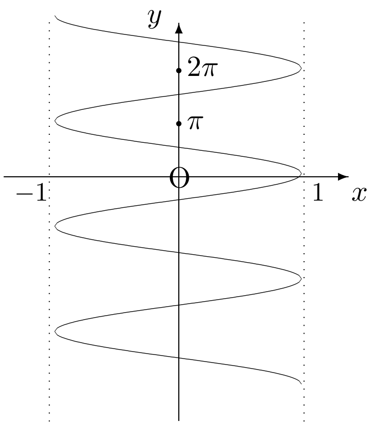
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \text{Cos}^{-1}x,$$

which may be obtained from the graph of $y = \cos x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

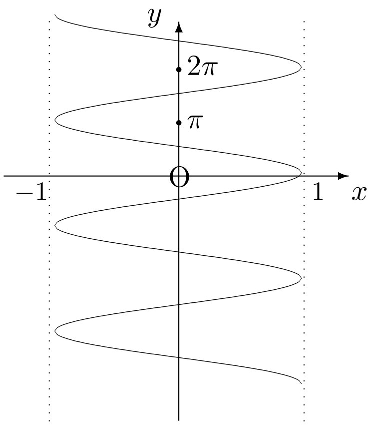


Observations

(i) x must lie in the interval $-1 \leq x \leq 1$.

(ii) For each x in $-1 \leq x \leq 1$, y has infinitely many values - spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one positive and the other negative.



(iv) On the part of the graph from $y = 0$ to $y = \pi$, we may distinguish the results from those of the inverse sine function.

There will be only one value of y with one (negative) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each x in $-1 \leq x \leq 1$.

The restricted part of the graph defines the **"principal value"** of the inverse cosine function and is denoted by $\cos^{-1}x$ using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}.$$

10.6.4 THE DERIVATIVE OF AN INVERSE TANGENT

We shall consider the formula

$$y = \mathrm{Tan}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case T in the formula (see later).

The formula is equivalent to

$$x = \tan y,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y \equiv 1 + \tan^2 y \equiv 1 + x^2.$$

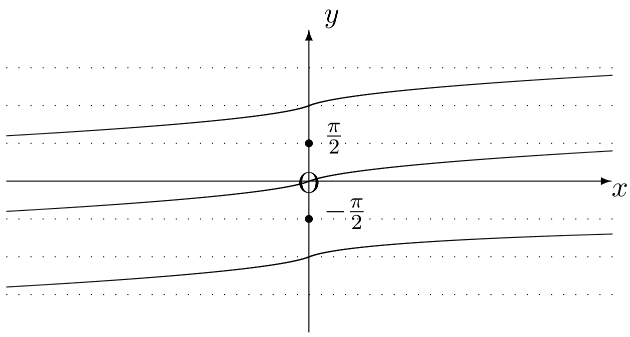
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 + x^2}.$$

Consider now the graph of the formula

$$y = \text{Tan}^{-1}x,$$

which may be obtained from the graph of $y = \tan x$ by reversing the roles of x and y and rearranging the new axes into the usual positions.

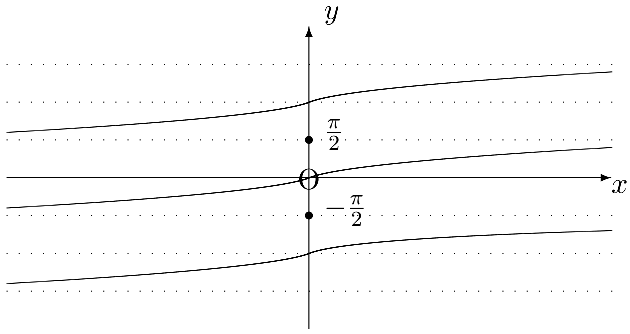


Observations

(i) x may lie anywhere in the interval $-\infty < x < \infty$.

(ii) For each x, y has infinitely many values - spaced at regular intervals of π.

(iii) For each x, there is only possible value of $\frac{dy}{dx}$, which is positive.



(iv) On the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y for each value of x.

The restricted part of the graph defines the **"principal value"** of the inverse tangent function and is denoted by $\tan^{-1}x$ using a lower-case t.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}.$$

ILLUSTRATIONS

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}2x] = \frac{2}{\sqrt{1-4x^2}}.$$

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}(x+3)] = -\frac{1}{\sqrt{1-(x+3)^2}}.$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}(\sin x)] = \frac{\cos x}{1 + \sin^2 x}.$$

4.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}\left(x^5\right)] = \frac{5x^4}{\sqrt{1 - x^{10}}}.$$