"JUST THE MATHS"

UNIT NUMBER

9.9

MATRICES 9
(Modal & spectral matrices)

by

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UNIT 9.9 - MATRICES 9

MODAL AND SPECTRAL MATRICES

9.9.1 ASSUMPTIONS AND DEFINITIONS

For convenience, we shall make, here, the following assumptions:

(a) The n eigenvalues, $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, of an $n \times n$ matrix, A, are arranged in order of decreasing value.

(b) Corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, respectively, A possesses a full set of eigenvectors $X_1,\ X_2,\ X_3, \ldots, X_n$, which are linearly independent.

If two eigenvalues coincide, the order of writing down the corresponding pair of eigenvectors will be immaterial.

DEFINITION 1

The square matrix obtained by using as its columns any set of linearly independent eigenvectors of a matrix A is called a **"modal matrix"** of A, and may be denoted by M.

Notes:

(i) There are infinitely many modal matrices for a given matrix, A, since any multiple of an eigenvector is also an eigenvector.

(ii) It is sometimes convenient to use a set of normalised eigenvectors.

When using normalised eigenvectors, the modal matrix may be denoted by N and, for an $n \times n$ matrix, A, there are 2^n possibilities for N since each of the n columns has two possibilities.

DEFINITION 2

If $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix, A, then the diagonal matrix,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \lambda_n \end{bmatrix}$$

is called the **"spectral matrix"** of A, and may be denoted by S.

EXAMPLE

For the matrix,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

determine a modal matrix, a modal matrix of normalised eigenvectors and the spectral matrix.

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0,$$

which may be shown to give

$$-(1+\lambda)(1-\lambda)(2-\lambda) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 2, \ \lambda_2 = 1$ and $\lambda_3 = -1$ in order of decreasing value.

Case 1. $\lambda = 2$

We solve the simultaneous equations

$$\begin{aligned} -x + y - 2z &= 0, \\ -x + 0y + z &= 0, \\ 0x + y - 3z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 3 : 1$

Case 2. $\lambda = 1$

We solve the simultaneous equations

$$\begin{aligned} 0x + y - 2z &= 0, \\ -x + y + z &= 0, \\ 0x + y - 2z &= 0, \end{aligned}$$

which give $x : y : z = 3 : 2 : 1$

Case 3. $\lambda = -1$

We solve the simultaneous equations

$$\begin{aligned} 2x + y - 2z &= 0, \\ -x + 3y + z &= 0, \\ 0x + y + 0z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 0 : 1$

A modal matrix for A may therefore be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

A modal matrix of normalised eigenvectors may be given by

$$\mathrm{N} = \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{2}{\sqrt{14}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9.9.2 DIAGONALISATION OF A MATRIX

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, it is clear that a matrix, A, and its spectral matrix, S, have the same eigenvalues.

From the Theorem in Unit 9.8, therefore, it seems reasonable that A and S could be similar matrices; and this is the content of the following result which will be illustrated rather than proven.

THEOREM

The matrix, A, is similar to its spectral matrix, S, the similarity transformation being

$$M^{-1}AM = S,$$

where M is a modal matrix for A.

ILLUSTRATION:

Suppose that $X_1,\ X_2$ and X_3 are linearly independent eigenvectors of a 3×3 matrix, A, corresponding to eigenvalues $\lambda_1,\ \lambda_2$ and λ_3, respectively.

Then,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad \text{and} \quad AX_3 = \lambda_3 X_3.$$

Also,

$$M = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

If M is premultiplied by A, we obtain a 3×3 matrix whose columns are $AX_1,\ AX_2,$ and AX_3.

That is,

$$AM = [\,AX_1 \quad AX_2 \quad AX_3\,] = [\,\lambda_1 X_1 \quad \lambda_2 X_2 \quad \lambda_3 X_3\,]$$

or

$$AM = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = MS.$$

We conclude that

$$M^{-1}AM = S.$$

Notes:

(i) M^{-1} exists only because X_1, X_2 and X_3 are linearly independent.

(ii) The similarity transformation in the above theorem reduces the matrix, A, to **"diagonal form"** or **"canonical form"** and the process is often referred to as the **"diagonalisation"** of the matrix, A.

EXAMPLE

Verify the above Theorem for the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution

From an earlier example, a modal matrix for A may be given by

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It may be shown that

$$M^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}$$

and, hence,

$$M^{-1}AM = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = S.$$

9.9.3 EXERCISES

1. Determine a modal matrix, M, of linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}.$$

   Verify that $M^{-1}AM = S$, where S is the spectral matrix of A.

2. Determine a modal matrix, M, of linearly independent eigenvectors for the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

   Verify that $M^{-1}AM = S$, where S is the spectral matrix of B.

3. Determine a modal matrix, N, of linearly independent normalised eigenvectors for the matrix

$$C = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}.$$

   Verify that $N^{-1}AN = S$, where S is the spectral matrix of C.

4. Show that the following matrices are not similar to a diagonal matrix:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

9.9.4 ANSWERS TO EXERCISES

1. The eigenvalues are 5,2 and −1, which gives

$$M = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}.$$

2. The eigenvalues are 2, 1 and −1, which gives

$$M = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

3. The eigenvalues are 4, 2 and 1, which gives

$$N = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{5}} & 1 \end{bmatrix}.$$

4. (a) The eigenvalues are 2 (repeated) and 1 but there are only two linearly independent eigenvectors, namely

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) There is only one eigenvalue, 1 (repeated), and only one linearly independent eigenvector, namely

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$