"JUST THE MATHS"

UNIT NUMBER

9.5

MATRICES 5
(Consistency and rank)

by

A.J.Hobson

UNIT 9.5 - MATRICES 5 - CONSISTENCY AND RANK

9.5.1 THE CONSISTENCY OF SIMULTANEOUS LINEAR EQUATIONS

Introduction

Methods of solving simultaneous linear equations in earlier Units have already shown that some sets of equations cannot be solved to give a unique solution. The Gaussian Elimination method described in Unit 9.4 is able to detect such situations as illustrated in the work below:

ILLUSTRATION 1.

Suppose, firstly, that we were required to investigate the solution of the simultaneous linear equations

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 7. \end{aligned}$$

The Gaussian Elimination solution, with check column, proceeds as follows

$$\begin{array}{ccc|c|c} 3 & -1 & 1 & 1 & 4 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 7 & 9 \end{array}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 7 & 9 \end{array}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 5R_1$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 16 & -34 & 2 & -16 \end{array}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 0 & 0 & 6 & 6 \end{array}$$

The third line seems to imply that $0.z = 6$; that is, $0 = 6$ which is impossible.

Hence the equations have no solution and are said to be **"inconsistent"**.

ILLUSTRATION 2.
Secondly, let us consider the simultaneous linear equations

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 1, \end{aligned}$$

which differ from the first illustration in one number only, the constant term of the third equation; but the conclusion will be very different.

This time, the Gaussian Elimination method gives the following sequence of steps:

3	-1	1	1	4
2	2	-5	0	-1
5	1	-4	1	3

$R_1 \rightarrow R_1 - R_2$

1	-3	6	1	5
2	2	-5	0	-1
5	1	-4	1	3

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 5R_1$

1	-3	6	1	5
0	8	-17	-2	-11
0	16	-34	-4	-22

$R_3 \rightarrow R_3 - 2R_2$

1	-3	6	1	5
0	8	-17	-2	-11
0	0	0	0	0

The third line here implies that the third equation is redundant since any set of x, y and z values would satisfy it. Hence, the original set of equations is equivalent to two equations in three unknowns, namely

$$\begin{aligned} x - 3y + 6z &= 1, \\ 8y - 17z &= -2. \end{aligned}$$

To state a suitable form of the conclusion, we could use the fact that any one of the three variables may be chosen at random, the other two being expressed in terms of it. For instance, if we choose z at random, then

$$x = \frac{3z+2}{8} \quad \text{and} \quad y = \frac{17z-2}{8}.$$

However, there is a neater way of arriving at the conclusion.

Neater form of solution

Suppose that $x = x_0$, $y = y_0$, $z = z_0$ is any **known** solution to the equations. It could be determined, for instance, by starting with $z = 0$; in this case, $z = 0$, $y = -\frac{1}{4}$ and $x = \frac{1}{4}$.

Let us now substitute

$$\begin{aligned} x &= x_1 + x_0, \\ y &= y_1 + y_0, \\ z &= z_1 + z_0, \end{aligned}$$

from which we obtain

$$\begin{aligned} (x_1 + x_0) - 3(y_1 + y_0) + 6(z_1 + z_0) &= 1, \\ 8(y_1 + y_0) - 17(z_1 + z_0) &= -2. \end{aligned}$$

But, because (x_0, y_0, z_0) is a known solution, this reduces to

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0. \end{aligned}$$

This is a set of **"homogeneous linear equations"** and, although clearly satisfied by $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, we regard this as a **"trivial solution"** and ignore it.

Of more use to us is the fact that an infinite number of non-trivial solutions can be found for each of which the variables x_1, y_1 and z_1 are in a certain set of ratios. In the present case, from the second equation, we have

$y_1 = \frac{17}{8}z_1$ which means that $y_1 : z_1 = 17 : 8$.

Substituting into the first equation gives

$x_1 - \frac{51}{8}z_1 + 6z_1 = 0$ which means that $x_1 = \frac{3}{8}z_1$; that is, $x_1 : z_1 = 3 : 8$.

Combining these conclusions, we can say that

$$x_1 : y_1 : z_1 = 3 : 17 : 8$$

and any three numbers in these ratios will serve as a set of values for x_1, y_1 and z_1.

The neater form of solution can, in general, be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$

where α may be any non-zero number.

In our present example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 17 \\ 8 \end{bmatrix}.$$

Notes:
(i) An alternative way of determining the ratios $x_1 : y_1 : z_1$ is to use the fact that, since at least one of the variables is going to be non-zero, we may begin by letting it equal 1.

In the above illustration, suppose we let $z_1 = 1$; then

$$y_1 = \frac{17}{8} \quad \text{and} \quad x_1 = \frac{3}{8}.$$

Hence,

$$x_1 : y_1 : z_1 = \frac{3}{8} : \frac{17}{8} : 1,$$

which can be rewritten

$$x_1 : y_1 : z_1 = 3 : 17 : 8,$$

as before.

(ii) Should it happen that a set of simultaneous linear equations reduces to **only one** equation (that is, each equation is just a multiple of the first) then a similar procedure can be applied as in the following illustration:

ILLUSTRATION 3.

Using trial and error, the equation

$$3x - 2y + 5z = 6$$

has a particular solution $x_0 = 1, y_0 = 1, z_0 = 1$, so that the general solution is given by

$$x = x_0 + x_1,\ y = y_0 + y_1,\ z = z_0 + z_1,$$

where

$$3x_1 - 2y_1 + 5z_1 = 0.$$

We could choose $x_1 = \alpha$ and $y_1 = \beta$ at random for this equation to give $z_1 = \frac{2\beta - 3\alpha}{5}$. That is,

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

9.5.2 THE ROW-ECHELON FORM OF A MATRIX

In the Gaussian Elimination method to solve a set of simltaneous linear equations,

$$MX = K,$$

we begin with the augmented matrix M|K and use elementary row operations to obtain more zeros at the beginning of each row than at the beginning of the previous row.

If desired, the first non-zero element in each row could be reduced to 1 by simply dividing that row throughout by the value of the first non-zero element. This leads to the following definition:

DEFINITION
The **"row echelon form"** of a matrix is that for which the first non-zero element in each row is 1 and occurs to the right of the first non-zero element in the previous row.

Note:
When using the term "row echelon form" in future, we shall not insist that the first non-zero element in each row has **actually** been reduced to 1.

9.5.3 THE RANK OF A MATRIX

The two illustrations of Gaussian Elimination discussed in section 9.5.1 may be used to imply that special conclusions are reached when, in the final row echelon form, either a complete row of M has reduced to zero (Illustration 1.) or a complete row of M|K has reduced to zero (Illustration 2.) This leads to another definition as follows:

DEFINITION
The **"rank"** of a matrix is the number or rows which do not reduce to a complete row of zeros when the matrix has been converted to row echelon form.

ILLUSTRATIONS

1. In our previous Illustration 1, M had rank 2 but M|K had rank 3. The equations were inconsistent.
2. In our previous Illustration 2, M had rank 2 and M|K also had rank 2. The equations had an infinite number of solutions.
3. In the examples of Unit 9.4 both M and M|K had rank 3. There was a unique solution to the simultaneous equations.

A general summary of these observations may be set out as follows:

1. The equations MX = K are inconsistent if rank M $<$ rank M|K.
2. The equations MX = K have an infinite number of solutions if rank M = rank M|K $< n$ where n is the number of equations.
3. The equations MX = K have a unique solution if rank M = rank M|K $= n$ where n is the number of equations.

9.5.4 EXERCISES

1. Use Gaussian Elimination to show that the following sets of simultaneous equations are inconsistent:

   (a)

$$\begin{aligned} x - y + 2z &= 2, \\ 3x + y - z &= 3, \\ 5x - y + 3z &= 4; \end{aligned}$$

   (b)

$$\begin{aligned} x - y + 2z &= 1, \\ -x + 3y - z &= -1, \\ 3x - 7y + 4z &= 5. \end{aligned}$$

2. Determine the rank of the following matrices:

   (a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 3 & 4 & 11 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} a & 1 & 2 \\ -1 & -3 & 8 \\ 1 & 12 & -3 \\ 4 & -3 & 7 \end{bmatrix}$.

3. Determine the general solution of the following equations by reducing the augmented matrix to row echelon form:

$$\begin{aligned} x + 3y - z &= 6, \\ 8x + 9y + 4z &= 21, \\ 2x + y + 2z &= 3. \end{aligned}$$

4. State the value of t for which the matrix

$$\mathrm{M} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & t & -6 \\ 3t & 3 & -9 \end{bmatrix}$$

has rank 1 and determine the general solution of the system of equations

$$\mathrm{MX} = \mathrm{K},$$

where

$$\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathrm{K} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

5. For the system of simultaneous linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1, \\ 2x_1 - x_2 + tx_3 &= 2, \\ -x_1 + 2x_2 + x_3 &= s, \end{aligned}$$

determine for which values of s and t there exists

(a) no solution;

(b) a unique solution;

(c) an infinite number of solutions.

Solve the equations for the two cases $s = 1, t = 1$ and $s = -1, t = 7$.

6. Determine the values of t for which the matrix

$$\mathrm{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & t \\ 3t & 2 & -2 \end{bmatrix}$$

has rank 2.

For each of these two values of t, solve the system of equations

$$\mathrm{MX} = \mathrm{K},$$

where $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathrm{K} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

9.5.5 ANSWERS TO EXERCISES

1. The rank of the matrix of x, y and z coefficients is less than the rank of the augmented matrix.

2. (a) 2; (b) 3.

3. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{9}{5} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -7 \\ 4 \\ 5 \end{bmatrix}$.

4. $$t = 2$$

   and

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}.$$

5. (a) $t = 7$ and $s \neq -1$; (b) $t \neq 7$; (c) $t = 7$ and $s = -1$.
   If $s = 1$ and $t = -1$, then $x_1 = \frac{7}{4}$, $x_2 = \frac{5}{4}$ and $x_3 = \frac{1}{4}$.
   If $s = -1$ and $t = 7$, then
   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}.$$

6. The rank is 2 when $t = -1$ or when $t = \frac{2}{3}$.
   If $t = -1$, the equations are inconsistent.
   If $t = \frac{2}{3}$, the equations have general solution
   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -5 \\ 8 \\ 3 \end{bmatrix}.$$