"JUST THE MATHS"

UNIT NUMBER

9.2

MATRICES 2
(Further matrix algebra)

by

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UNIT 9.2 - MATRICES 2 - THE ALGEBRA OF MATRICES (Part Two)

9.2.1 MULTIPLICATION BY A SINGLE NUMBER

If we were required to multiply a matrix of any order by a **positive whole number**, n, we would clearly regard the operation as equivalent to adding together n copies of the given matrix. Thus, in the result, every element of this given matrix would be multiplied by n; but it is logical to extend the idea to the multiplication of a matrix by **any** number, λ, not necessarily a positive whole number, the rule being to multiply every element of the matrix by λ.

In symbols we could say that

$$\lambda \left[a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n},$$

where

$$b_{ij} = \lambda a_{ij}.$$

Note:
The rule for multiplying a matrix by a single number can also be used in reverse to remove common factors from the elements of a matrix as illustrated as follows:

ILLUSTRATION

$$\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

9.2.2 THE PRODUCT OF TWO MATRICES

The definition of a matrix product is more difficult to justify than the previous concepts, partly because it is by no means an obvious definition and partly because we cannot be sure exactly what originally led to the making of the definition.

Some hint is given by the matrix equation at the end of the introduction to Unit 9.1, where the product of a 2×2 matrix and a 2×1 matrix is another 2×1 matrix; but we must be prepared to meet other orders of matrix as well.

We shall introduce the definition with a semi-practical illustration, then make a formal statement of the definition itself.

ILLUSTRATION

A motor manufacturer, with three separate factories, makes two types of car, one called "standard" and the other called "luxury".
In order to manufacture each type of car, he needs a certain number of units of material and a certain number of units of labour each unit representing £300.

A table of data to represent this information could be

Type	Materials	Labour
Standard	12	15
Luxury	16	20

The manufacturer receives an order from another country to supply 400 standard cars and 900 luxury cars; but he distributes the export order amongst his three factories as follows:

Location	Standard	Luxury
Factory A	100	400
Factory B	200	200
Factory C	100	300

The number of units of material and the number of units of labour needed by each factory to complete the order may be given by another table, namely

Location	Materials	Labour
Factory A	$100 \times 12 + 400 \times 16$	$100 \times 15 + 400 \times 20$
Factory B	$200 \times 12 + 200 \times 16$	$200 \times 15 + 200 \times 20$
Factory C	$100 \times 12 + 300 \times 16$	$100 \times 15 + 300 \times 20$

If we now replace each table by the corresponding matrix, the calculations appear as the product of a 3×2 matrix and a 2×2 matrix. That is,

$$\underset{3 \times 2}{\begin{bmatrix} 100 & 400 \\ 200 & 200 \\ 100 & 300 \end{bmatrix}} \cdot \underset{2 \times 2}{\begin{bmatrix} 12 & 15 \\ 16 & 20 \end{bmatrix}} = \underset{3 \times 2}{\begin{bmatrix} 100 \times 12 + 400 \times 16 & 100 \times 15 + 400 \times 20 \\ 200 \times 12 + 200 \times 16 & 200 \times 15 + 200 \times 20 \\ 100 \times 12 + 300 \times 16 & 100 \times 15 + 300 \times 20 \end{bmatrix}} = \underset{3 \times 2}{\begin{bmatrix} 7600 & 9500 \\ 5600 & 7000 \\ 6000 & 7500 \end{bmatrix}}$$

OBSERVATIONS

(i) The product matrix has 3 rows because the first matrix on the left has 3 rows.

(ii) The product matrix has 2 columns because the second matrix on the left has 2 columns.

(iii) The product cannot be worked out unless the number of columns in the first matrix matches the number of rows in the second matrix with no elements left over in the pairing-up process.

(iv) The elements of the product matrix are systematically obtained by multiplying (in pairs) the corresponding elements of each row in the first matrix with each column in the second matrix. To pair up the correct elements, we read each row of the first matrix from left to right and each column of the second matrix from top to bottom.

The Formal Definition of a Matrix Product
If A and B are matrices, then the product AB is defined (that is, it has a meaning) only when the number of columns in A is equal to the number of rows in B.

If A is of order $m \times n$ and B is of order $n \times p$, then AB is of order $m \times p$.

To obtain the element in the i-th row and j-th column of AB, we multiply corresponding elements of the i-th row of A and the j-th column of B, then add up the results.

ILLUSTRATION

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 1 & -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -1 & 12 \\ 1 & -10 & 1 & 15 \end{bmatrix}$$

Note:
Confusion could arise when multiplying a matrix of order 1×1 by another matrix. Apparently, the other matrix would need to have either a single row or a single column depending on the order of multiplication.

However, as stated in Unit 9.1, a matrix of order 1×1 is considered to be a special case, and is defined separately to be the same as a single number. Hence a matrix of any order can be multiplied by a matrix of order 1×1 even though this does not fit the formal rules for matrix multiplication in general.

9.2.3 THE NON-COMMUTATIVITY OF MATRIX PRODUCTS

In elementary arithmetic, if a and b are two numbers, then $ab = ba$ (that is, the product **"commutes"**). But this is not so for matrices A and B as we now show:

(a) If A is of order $m \times n$, then B must be of order $n \times m$ if both AB **and** BA are to be defined.

(b) AB and BA will have different orders unless $m = n$, in which case the two products will be square matrices of order $m \times m$.

(c) Even if A and B are **both** square matrices of order $m \times m$, it will not normally be the case that AB is the same as BA. A simple numerical example will illustrate this fact:

$$\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 11 & 35 \end{bmatrix}; \quad \text{but} \quad \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 19 & 37 \end{bmatrix}.$$

Notes:
(i) If we simply wanted to show that AB $\neq$ BA, we would need only to demonstrate that one pair of corresponding elements were unequal in value.

(ii) If such a basic rule of elementary arithmetic is false for matrices, we should, strictly speaking, be prepared to justify other basic rules of arithmetic. But it turns out that the non-commutativity of matrix products is the only one which causes problems.

For instance, it can be shown that, provided the matrices involved are compatible for addition or multiplication,

$\mathrm{A} + \mathrm{B} \equiv \mathrm{B} + \mathrm{A}$; the **"Commutative Law of Addition"**.

$\mathrm{A} + (\mathrm{B} + \mathrm{C}) \equiv (\mathrm{A} + \mathrm{B}) + \mathrm{C}$; the **"Associative Law of Addition"**.

$\mathrm{A}(\mathrm{BC}) \equiv (\mathrm{AB})\mathrm{C}$; the **"Associative Law of Multiplication"**.

$\mathrm{A}(\mathrm{B} + \mathrm{C}) \equiv \mathrm{AB} + \mathrm{BC}$ or $(\mathrm{A} + \mathrm{B})\mathrm{C} \equiv \mathrm{AC} + \mathrm{BC}$; the **"Distributive Laws"**.

(iii) In the matrix product, AB, we say either that B is **"pre-multiplied"** by A or that A is **"post-multiplied"** by B.

9.2.4 MULTIPLICATIVE IDENTITY MATRICES

In connection with matrix multiplication and, subsequently, the solution of simultaneous linear equations, an important type of matrix is a square matrix with the number 1 in each position of the leading diagonal, but zeros everywhere else. Such a matrix is denoted by I_n if there are n rows and (of course) n columns. If it is not necessary to specify the number of rows and columns, the notation I, without a subscript, is sufficient.

ILLUSTRATIONS

$$\mathrm{I}_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathrm{I}_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{I}_4 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Any matrix of the type I_n multiplies another matrix (with an appropriate number of rows or columns) to leave it identically the same as it was to start with. For this reason, I_n is called a **"multiplicative identity matrix"**, although we normally call it just an "identity matrix" (unless it becomes necessary to distinguish it from the **additive** identity matrix referred to earlier). Another common name for it is a **"unit matrix"**.

For example, suppose that

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Then, post-multiplying by I_2, it is easily checked that

$$\mathrm{AI}_2 = \mathrm{A}.$$

Similarly, pre-multiplying by I_3, it is easily checked that

$$\mathrm{I}_3\mathrm{A} = \mathrm{A}.$$

In general, if A is of order $m \times n$, then

$$\mathrm{AI}_n = \mathrm{I}_m\mathrm{A} = \mathrm{A}.$$

9.2.5 EXERCISES

1. If $\mathrm{A} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ and $\mathrm{C} = \begin{bmatrix} -1 & 4 & -2 \\ 0 & 1 & 0 \end{bmatrix}$,
   determine the elements of the following matrices:
   (a) A + 2B; (b) A + 2B - 3C; (c) $3\mathrm{A}^\mathrm{T} - \mathrm{B}^\mathrm{T}$.
2. Remove a common factor from each of the following matrices in order to express it as the product of a number and a matrix:
   (a) $\begin{bmatrix} 8 & -4 \\ -32 & 16 \end{bmatrix}$; (b) $\begin{bmatrix} -x^3 & -x^2 \\ x^2 & -4x^2 \end{bmatrix}$.
3. State the order of the product matrix in each of the following cases:
   (a) $\mathrm{A}_{1\times2}.\mathrm{B}_{2\times2}$; (b) $\mathrm{A}_{3\times1}.\mathrm{B}_{1\times2}$; (c) $\mathrm{A}_{4\times3}.\mathrm{B}_{3\times5}$.
4. For the matrices $\mathrm{A}_{2\times2}, \mathrm{B}_{2\times3}, \mathrm{C}_{3\times3}$ and $\mathrm{D}_{2\times4}$, which of the following products are defined.
   (a) A.B; (b) B.C; (c) C.D; (d) A.C; (e) A.D.

5. Determine the elements of the product matrix in each of the following:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 1 \end{bmatrix} . \begin{bmatrix} 12 & 24 \\ 24 & 36 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 4 & -3 \\ 1 & -1 & 0 \end{bmatrix} . \begin{bmatrix} 2 & 6 \\ 5 & -3 \\ -1 & 7 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} . \begin{bmatrix} 7 & -3 \\ -1 & 4 \end{bmatrix}$; (e) $[4 \quad 2 \quad 1] . \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$; (f) $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} . [4 \quad 2 \quad 1]$;

(g) $[-1 \quad 2] . \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (h) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & 9 \end{bmatrix} . \begin{bmatrix} -1 & 3 & -2 \\ -0.5 & -3 & 2.5 \\ 1 & 1 & -1 \end{bmatrix}$.

6. If $\mathrm{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $\mathrm{A.A^T} = \mathrm{I_2}$.

7. If $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, verify that $\mathrm{(A.B)^T} = \mathrm{B^T.A^T}$ **not** $\mathrm{A^T.B^T}$.

9.2.6 ANSWERS TO EXERCISES

1. (a) $\begin{bmatrix} 9 & 3 & 6 \\ 2 & 0 & 5 \end{bmatrix}$; (b) $\begin{bmatrix} 12 & -9 & 12 \\ 2 & -3 & 5 \end{bmatrix}$; (c) $\begin{bmatrix} -1 & 13 \\ -12 & 0 \\ 4 & 1 \end{bmatrix}$.

2. (a) $4. \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$; (b) $-x^2. \begin{bmatrix} x & 1 \\ -1 & 4 \end{bmatrix}$.

3. (a) 1×2; (b) 3×2; (c) 4×5.

4. (a), (b) and (e) are defined.

5. (a) $\begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$; (b) $\begin{bmatrix} 14 & 24 \\ 27 & 42 \end{bmatrix}$; (c) $\begin{bmatrix} 23 & -33 \\ -3 & 9 \end{bmatrix}$;

(d) $\begin{bmatrix} 5 & 5 \\ 21 & -9 \\ 3 & 13 \end{bmatrix}$; (e) [4] defined as 4; (f) $\begin{bmatrix} -4 & -2 & -1 \\ 12 & 6 & 3 \\ 8 & 4 & 2 \end{bmatrix}$;

(g) $[-1 \quad 2 \quad -1]$; (h) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.