"JUST THE MATHS"

UNIT NUMBER

9.10

MATRICES 10
(Symmetric matrices & quadratic forms)

by

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UNIT 9.10 - MATRICES 10

SYMMETRIC MATRICES AND QUADRATIC FORMS

9.10.1 SYMMETRIC MATRICES

The definition of a symmetric matrix was introduced in Unit 9.1 and matrices of this type have certain special properties with regard to eigenvalues and eigenvectors. We list them as follows:

(i) All of the eigenvalues of a symmetric matrix are real and, hence, so are the eigenvectors.

(ii) A symmetric matrix of order $n \times n$ always has n linearly independent eigenvectors.

(iii) For a symmetric matrix, suppose that X_i and X_j are linearly independent eigenvectors associated with different eigenvectors; then

$$\mathrm{X}_i \mathrm{X}_j^{\mathrm{T}} \equiv x_i x_j + y_i y_j + z_i z_j = 0.$$

We say that X_i and X_j are **"mutually orthogonal"**.

If a symmetric matrix has any repeated eigenvalues, it is still possible to determine a full set of mutually orthogonal eigenvectors, but not every full set of eigenvectors will have the orthogonality property.

(iv) A symmetric matrix always has a modal matrix whose columns are mutually orthogonal. When the eigenvalues are distinct, this is true for every modal matrix.

(v) A modal matrix, N, of normalised eigenvectors is an orthogonal matrix.

ILLUSTRATIONS

1. If N is of order 3×3, we have

$$\mathrm{N}^{\mathrm{T}}.N = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} . \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. It was shown in Unit 9.6 that the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$, and $\lambda = -1$ (repeated), with associated eigenvectors

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} -\frac{1}{2}\beta - \gamma \\ \beta \\ \gamma \end{bmatrix}.$$

A set of **linearly independent** eigenvectors may therefore be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, X_1 is orthogonal to X_2 and X_3, but X_2 and X_3 are not orthogonal to each other. However, we may find β and γ such that

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ is orthogonal to } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We simply require that

$$\frac{1}{2}\beta + 2\gamma = 0$$

or

$$\beta + 4\gamma = 0;$$

and this will be so, for example, when $\beta = 4$ and $\gamma = -1$.

A new set of linearly independent mutually orthogonal eigenvectors can thus be given by

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

9.10.2 QUADRATIC FORMS

An algebraic expression of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2yzx + 2hxy$$

is called a **"quadratic form"**.

In matrix notation, it may be written as

$$[\,x \quad y \quad z\,] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv X^T A X,$$

and we note that the matrix A is symmetric.

In the scientific applications of quadratic forms, it is desirable to know whether such a form is

(a) always positive,

(b) always negative,

(c) both positive and negative.

It may be shown that, if we change to new variables, (u, v, w), using a linear transformation

$$X = PU,$$

where P is some non-singular matrix, then the new quadratic form has the same properties as the original, concerning its sign.

We now show that a good choice for P is a modal matrix, N, of normalised, linearly independent, mutually orthogonal eigenvectors for A.

Putting X = NU, the expression X^TAX becomes U^TN^TANX.

But, since N is orthogonal when A is symmetric, $N^T = N^{-1}$ and, hence, N^TAN is the spectral matrix, S, for A.

The new quadratic form is therefore

$$U^TSU \equiv [\,u \quad v \quad w\,] . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2.$$

Clearly, if all of the eigenvalues are positive, then the new quadratic form is always positive; and, if all of the eigenvalues are negative, then the new quadratic form is always negative.

The new quadratic form is called the **"canonical form under similarity"** of the original quadratic form.

9.10.3 EXERCISES

1. For the following symmetric matrices, determine a set of three linearly independent and mutually orthogonal eigenvectors:

$$\text{(a)} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 6 \\ 0 & 6 & 5 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Repeat the previous question for the following symmetric matrices:

$$\text{(a)} \quad \begin{bmatrix} 3 & 3 & 3\sqrt{2} \\ 3 & 3 & 3\sqrt{2} \\ 3\sqrt{2} & 3\sqrt{2} & 6 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix}.$$

3. Using the results of question 1, show that the following quadratic forms are always positive:

   (a)

$$2x^2 + 10y^2 + 5z^2 + 12yz;$$

   (b)

$$2x^2 + 5y^2 + 3z^2 + 4xy.$$

4. Using the results of question 2(b), obtain the matrix, P, of the orthogonal transformation, X = PU, which transforms the quadratic function

$$2x^2 + y^2 - 2z^2 + 4xz$$

into the quadratic function

$$2u^2 + 2v^2 - 3w^2.$$

State whether or the not the original quadratic form is always positive.

9.10.4 ANSWERS TO EXERCISES

1. (a) The eigenvalues are 14, 2 and 1 and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}.$$

(b) The eigenvalues are 6, 3 and 1 and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

2. (a) The eigenvalues are 12 and 0 (repeated) and set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}.$$

(b) The eigenvalues are 2 and −3 (repeated) and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

3. The eigenvalues are all positive and hence the quadratic forms are always positive.

4.

$$\mathrm{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix}.$$

The original quadratic form may take both positive and negative values since the associated eigenvalues are not all positive.