"JUST THE MATHS"

UNIT NUMBER

8.6

VECTORS 6
(Vector equations of planes)

by

A.J.Hobson

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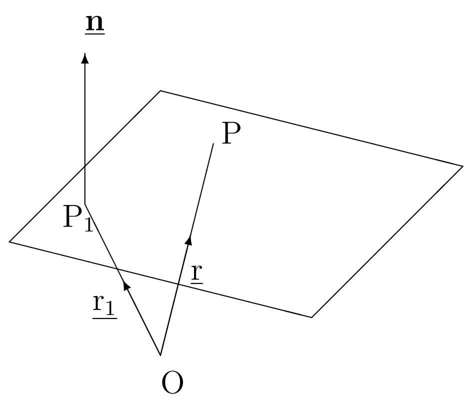
UNIT 8.6 - VECTORS 6

VECTOR EQUATIONS OF PLANES

8.6.1 THE PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

A plane in space is completely specified if we know one point in it, together with a vector which is perpendicular to the plane.



In the diagram, the given point is P_1, with position vector, r_1, and the given vector is $\underline{n}$.

Hence, the vector, $\underline{P_1P}$, is perpendicular to $\underline{n}$, which leads to the equation

$$(\underline{r} - \underline{r_1}) \bullet \underline{n} = 0$$

or

$$\underline{r} \bullet \underline{n} = \underline{r_1} \bullet \underline{n} = d \text{ say.}$$

Notes:

(i) In the particular case when $\underline{n}$ is a unit vector, the constant, d, represents the perpendicular projection of $\underline{r_1}$ onto $\underline{n}$, which is therefore the perpendicular distance of the origin from the plane.

(ii) If $\underline{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\underline{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the cartesian form for the equation of the above plane will be

$$ax + by + cz = d.$$

That is, it is simply a linear equation in the variables x, y and z.

EXAMPLE

Determine the vector equation and, hence, the cartesian equation of the plane passing through the point with position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and perpendicular to the vector $\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

Solution

The vector equation is

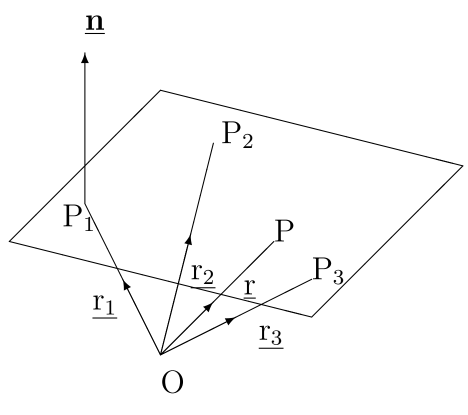
$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = (3)(1) + (-2)(-4) + (1)(-1) = 10$$

and, hence, the cartesian equation is

$$x - 4y - z = 10.$$

8.6.2 THE PLANE PASSING THROUGH THREE GIVEN POINTS

We consider a plane passing through the points, $\mathrm{P}_1(x_1, y_1, z_1)$, $\mathrm{P}_2(x_2, y_2, z_2)$ and $\mathrm{P}_3(x_3, y_3, z_3)$.



In the diagram, a suitable vector for $\underline{\mathrm{n}}$ is

$$\underline{\mathrm{P_1P_2}}\ \mathbf{x}\ \underline{\mathrm{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

and, hence, the equation,

$$(\underline{\mathrm{r}} - \underline{\mathrm{r_1}}) \bullet \underline{\mathrm{n}} = 0,$$

of the plane becomes

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

But, from the properties of determinants, this is equivalent to

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

which is the standard equation of the plane through the three given points.

EXAMPLE

Determine the cartesian equation of the plane passing through the three points, $(0, 2, -1)$, $(3, 0, 1)$ and $(-3, -2, 0)$.

Solution

The equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{vmatrix} = 0,$$

which, on expansion and simplification, gives

$$2x - 3y - 6z = 0,$$

showing that the plane also passes through the origin.

8.6.3 THE POINT OF INTERSECTION OF A STRAIGHT LINE AND A PLANE

First, we recall (from Unit 8.5) that the vector equation of a straight line passing through the fixed point, with position vector $\underline{\mathrm{r}_1}$, and parallel to the fixed vector $\underline{\mathrm{a}}$, is

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}}.$$

For the point of intersection of this line with the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d,$$

the value of t must be such that

$$(\underline{\mathrm{r}_1} + t\underline{\mathrm{a}}) \bullet \underline{\mathrm{n}} = d,$$

which is an equation from which the appropriate value of t and, hence, the point of intersection may be found.

EXAMPLE

Determine the point of intersection of the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7,$$

and the straight line passing through the point, $(4, -1, 3)$, which is parallel to the vector $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

Solution

We need to obtain the value of the parameter, t, such that

$$(4\mathbf{i} - \mathbf{j} + 3\mathbf{k} + t[2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}]) \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7.$$

That is,

$$(4+2t)(1)+(-1-2t)(-3)+(3+5t)(-1)=7 \quad \text{or} \quad 4+3t=7,$$

which gives $t=1$; and, hence, the point of intersection is $(4+2,-1-2,3+5)=(6,-3,8)$.

8.6.4 THE LINE OF INTERSECTION OF TWO PLANES

Suppose we are given two non-parallel planes whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1}=d_1 \quad \text{and} \quad \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2}=d_2.$$

Their line of intersection will be perpendicular to both $\underline{\mathrm{n}_1}$ and $\underline{\mathrm{n}_2}$, since these are the normals to the two planes.

The line of intersection will thus be parallel to $\underline{\mathrm{n}_1}$ **x** $\underline{\mathrm{n}_2}$, and all it remains to do, to obtain the vector equation of this line, is to determine any point on it.

For convenience, we may take the point (common to both planes) for which one of x, y or z is zero.

EXAMPLE

Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1}=2 \quad \text{and} \quad \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2}=17,$$

where

$$\underline{\mathrm{n}_1}=\mathbf{i}+\mathbf{j}+\mathbf{k} \quad \text{and} \quad \underline{\mathrm{n}_2}=4\mathbf{i}+\mathbf{j}+2\mathbf{k}.$$

Solution

Firstly,

$$\underline{\mathrm{n}_1} \mathbf{\ x\ } \underline{\mathrm{n}_2}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}.$$

Secondly, the cartesian equations of the two planes are

$$x + y + z = 2 \text{ and } 4x + y + 2z = 17;$$

and, when $z = 0$, these become

$$x + y = 2 \text{ and } 4x + y = 17,$$

which, as simultaneous linear equations, have a common solution of $x = 5$, $y = -3$.

Thirdly, therefore, a point on the line of intersection is $(5, -3, 0)$, which has position vector $5\mathbf{i} - 3\mathbf{j}$.

Hence, the vector equation of the line of intersection is

$$\underline{\mathrm{r}} = 5\mathbf{i} - 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}).$$

Finally, since $x = 5 + t$, $y = -3 + 2t$ and $z = -3t$ the line of intersection is represented, in standard cartesian form, by

$$\frac{x - 5}{1} = \frac{y + 3}{2} = \frac{z}{-3} \quad (= t).$$

8.6.5 THE PERPENDICULAR DISTANCE OF A POINT FROM A PLANE

Given the plane whose vector equation is $\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d$ and the point, P_1, whose position vector is $\underline{\mathrm{r}_1}$, the straight line through the point, P_1, which is perpendicular to the plane has vector equation

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{n}}.$$

This line meets the plane at the point, P_0, with position vector $\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}$, where

$$(\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) \bullet \underline{\mathrm{n}} = d.$$

That is,

$$(\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}}) + t_0\mathrm{n}^2 = d.$$

Hence,

$$t_0 = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}^2}$$

Finally, the vector $\underline{\mathrm{P}_0\mathrm{P}_1} = (\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) - \underline{\mathrm{r}_1} = t_0\underline{\mathrm{n}}$

and its magnitude, $t_0\mathrm{n}$, will be the perpendicular distance, p, of the point P_1 from the plane.

In other words,

$$p = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}}.$$

Note:
In terms of cartesian co-ordinates, this formula is equivalent to

$$p = \frac{d - (ax_1 + by_1 + cz_1)}{\sqrt{a^2 + b^2 + c^2}},$$

where a, b and c are the **i**, **j** and **k** components of $\underline{\mathrm{n}}$ respectively.

EXAMPLE

Determine the perpendicular distance, p, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

Solution

From the cartesian formula

$$p = \frac{13 - [(1)(2) + (2)(-3) + (2)(4)]}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

8.6.6 EXERCISES

1. Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector $\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, and perpendicular to the vector $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

2. Determine the cartesian equation of the plane passing through the three points $(1, -1, 2)$, $(3, -2, -1)$ and $(-1, 4, 0)$.

3. Determine the point of intersection of the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet (5\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -3,$$

and the straight line passing through the point $(2, 1, -3)$, which is parallel to the vector $\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

4. Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes, whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1} = 14 \quad \text{and} \quad \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2} = -1,$$

where

$$\underline{\mathrm{n}_1} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{\mathrm{n}_2} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

5. Determine, in surd form, the perpendicular distance of the point $(-5, -2, 8)$ from the plane whose cartesian equation is $2x - y + 3z = 17$.

8.6.7 ANSWERS TO EXERCISES

1.

$$\underline{\mathrm{r}} \bullet (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 10, \quad \text{or} \quad 2x + y - 3z = 10.$$

2.

$$17x + 10y + 8z = 23.$$

3.

$$(0, -1, 5).$$

4.

$$\underline{\mathrm{r}} = -2\mathbf{i} + 3\mathbf{j} + t(7\mathbf{i} + 10\mathbf{j} - 8\mathbf{k})$$

or

$$\frac{x+2}{7} = \frac{y-3}{10} = \frac{z}{-8} \quad (= t).$$

5.

$$\frac{1}{\sqrt{14}}.$$