"JUST THE MATHS"

UNIT NUMBER

8.5

VECTORS 5
(Vector equations of straight lines)

by

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UNIT 8.5 - VECTORS 5

VECTOR EQUATIONS OF STRAIGHT LINES

8.5.1 INTRODUCTION

The concept of vector notation and vector products provides a convenient method of representing straight lines and planes in space by simple vector equations. Such vector equations may then, if necessary, be converted back to conventional cartesian or parametric equations.

We shall assume that the position vector of a variable point, $\mathrm{P}(x, y, z)$, is given by

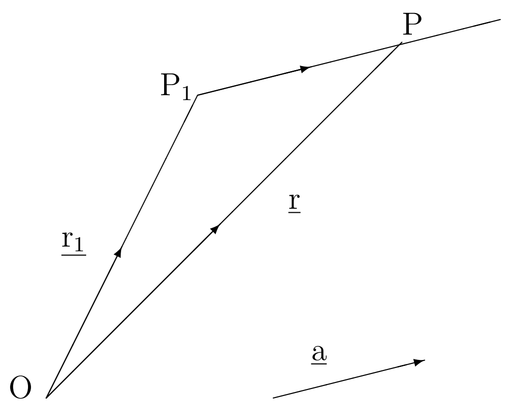
$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and that the position vectors of fixed points, such as $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$, are given by

$$\underline{\mathrm{r}_1} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \quad \underline{\mathrm{r}_2} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}, \quad \text{etc.}$$

8.5.2 THE STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

We consider, here, the straight line passing through the point, P_1, with position vector, $\underline{\mathrm{r}_1}$, and parallel to the vector, $\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.



From the diagram,

$$\underline{\mathrm{OP}} = \underline{\mathrm{OP}_1} + \underline{\mathrm{P}_1\mathrm{P}}.$$

But,

$$\underline{\mathrm{P}_1\mathrm{P}} = t\underline{\mathrm{a}},$$

for some number t.

Hence,

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}},$$

which is the vector equation of the straight line.

The components of $\underline{\mathrm{a}}$ form a set of direction ratios for the straight line.

Notes:

(i) The vector equation of a straight line passing through the origin and parallel to a given vector $\underline{\mathrm{a}}$ will be of the form

$$\underline{r} = t\underline{\mathrm{a}}.$$

(ii) By equating **i**, **j** and **k** components on both sides, the vector equation of the straight line passing through P_1 and parallel to $\underline{\mathrm{a}}$ leads to parametric equations

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t;$$

and, if these are solved for the parameter, t, we obtain

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3},$$

which is the standard cartesian form of the straight line.

EXAMPLES

1. Determine the vector equation of the straight line passing through the point with position vector $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and parallel to the vector, $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

   Express the vector equation of the straight line in standard cartesian form.

   Solution

   The vector equation of the straight line is

$$\underline{\mathrm{r}} = \mathbf{i} - 3\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

   or

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (1 + 2t)\mathbf{i} + (-3 + 3t)\mathbf{j} + (1 - 4t)\mathbf{k}.$$

   Eliminating t from each component, we obtain the cartesian form of the straight line,

$$\frac{x-1}{2} = \frac{y+3}{3} = \frac{z-1}{-4}.$$

2. The equations

$$\frac{3x+1}{2} = \frac{y-1}{2} = \frac{-z+5}{3}$$

   determine a straight line. Express them in vector form and obtain a set of direction ratios for the straight line.

   Solution

   Rewriting the equations so that the coefficients of x, y and z are unity, we have

$$\frac{x+\frac{1}{3}}{\frac{2}{3}} = \frac{y-1}{2} = \frac{z-5}{-3}.$$

   Hence, in vector form, the equation of the line is

$$\underline{\mathrm{r}} = -\frac{1}{3}\mathbf{i} + \mathbf{j} + 5\mathbf{k} + t\left(\frac{2}{3}\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\right).$$

   Thus, a set of direction ratios for the straight line are $\frac{2}{3} : 2 : -3$ or $2 : 6 : -9$.

3. Show that the two straight lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \text{ and } \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = \mathbf{j}, \quad \underline{\mathrm{a}_1} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

and

$$\underline{\mathrm{r}_2} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \underline{\mathrm{a}_2} = -2\mathbf{i} - 2\mathbf{j},$$

have a common point and determine its co-ordinates.

Solution

Any point on the first line is such that

$$x = t, \quad y = 1 + 2t, \quad z = -t,$$

for some parameter value, t; and any point on the second line is such that

$$x = 1 - 2s, \quad y = 1 - 2s, \quad z = 1,$$

for some parameter value, s.

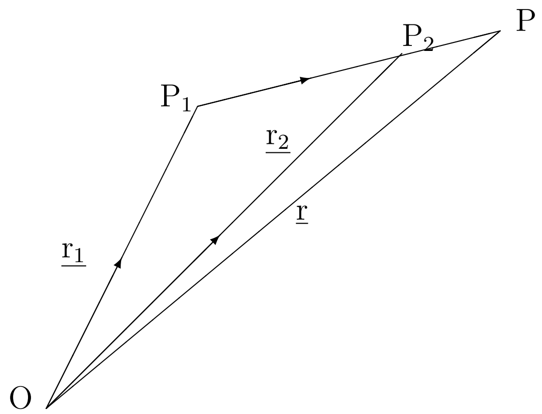
The lines have a common point if it is possible to find values of t and s such these are the same point.

In fact, $t = -1$ and $s = 1$ are suitable values and give the common point $(-1, -1, 1)$.

8.5.3 THE STRAIGHT LINE PASSING THROUGH TWO GIVEN POINTS

If a straight line passes through the two given points, P_1 and P_2, then it is certainly parallel to the vector,

$$\underline{\mathrm{a}} = \underline{\mathrm{P}_1\mathrm{P}_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$



Thus, the vector equation of the straight line is

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}}$$

as before.

Notes:

(i) The parametric equations of the straight line passing through the points, P_1 and P_2, are

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t;$$

and we notice that the "base-points" of the parametric representation (that is, P_1 and P_2) have parameter values $t = 0$ and $t = 1$ respectively.

(ii) The standard cartesian form of the straight line passing through P_1 and P_2 is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

EXAMPLE

Determine the vector equation of the straight line passing through the two points, $P_1(3, -1, 5)$ and $P_2(-1, -4, 2)$.

Solution

$$\underline{OP_1} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

and

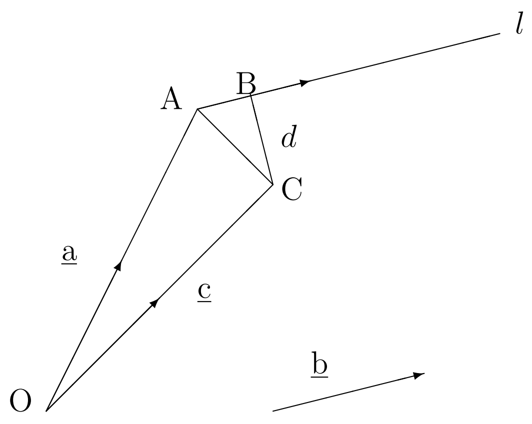
$$\underline{P_1P_2} = (-1-3)\mathbf{i} + (-4+1)\mathbf{j} + (2-5)\mathbf{k} = -4\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}.$$

Hence, the vector equation of the straight line is

$$\underline{r} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} - t(4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}).$$

8.5.4 THE PERPENDICULAR DISTANCE OF A POINT FROM A STRAIGHT LINE

For a straight line, l, passing through a given point, A, with position vector, $\underline{a}$ and parallel to a given vector, $\underline{b}$, it may be necessary to determine the perpendicular distance, d, from this line, of a point, C, with position vector, $\underline{c}$.



From the diagram, with Pythagoras' Theorem,

$$d^2 = (\text{AC})^2 - (\text{AB})^2.$$

But, $\underline{\text{AC}} = \underline{\text{c}} - \underline{\text{a}}$, so that

$$(\text{AC})^2 = (\underline{\text{c}} - \underline{\text{a}}) \bullet (\underline{\text{c}} - \underline{\text{a}}).$$

Also, the length, AB, is the projection of $\underline{\text{AC}}$ onto the line, l, which is parallel to $\underline{\text{b}}$.

Hence,

$$\text{AB} = \frac{(\underline{\text{c}} - \underline{\text{a}}) \bullet \underline{\text{b}}}{\text{b}},$$

which gives the result

$$d^2 = (\underline{\text{c}} - \underline{\text{a}}) \bullet (\underline{\text{c}} - \underline{\text{a}}) - \left[\frac{(\underline{\text{c}} - \underline{\text{a}}) \bullet \underline{\text{b}}}{\text{b}}\right]^2.$$

From this result, d may be deduced.

EXAMPLE

Determine the perpendicular distance of the point $(3, -1, 7)$ from the straight line passing through the two points, $(2, 2, -1)$ and $(0, 3, 5)$.

Solution

In the standard formula, we have

$$\underline{\text{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{j},$$

$$\underline{b} = (0-2)\mathbf{i} + (3-2)\mathbf{j} + (5-[-1])\mathbf{k} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k},$$

$$\text{b} = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41},$$

$$\underline{c} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k},$$

and

$$\underline{c} - \underline{a} = (3-2)\mathbf{i} + (-1-2)\mathbf{j} + (7-[-1])\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

Hence, the perpendicular distance, d, is given by

$$d^2 = 1^2 + (-3)^2 + 8^2 - \frac{(1)(-2) + (-3)(1) + (8)(6)}{\sqrt{41}} = 74 - \frac{43}{\sqrt{41}}$$

which gives $d \simeq 8.20$.

8.5.5 THE SHORTEST DISTANCE BETWEEN TWO PARALLEL STRAIGHT LINES

The result of the previous section may also be used to determine the shortest distance between two parallel straight lines, because this will be the perpendicular distance from one of the lines of any point on the other line.

We may consider the perpendicular distance between

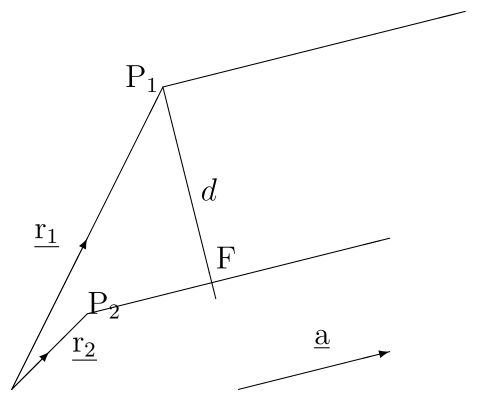
(a) the straight line passing through the fixed point with position vector $\underline{r_1}$ and parallel to the fixed vector, $\underline{a}$

and

(b) the straight line passing through the fixed point with position vector $\underline{r_2}$, also parallel to the fixed vector, $\underline{a}$.

These will have vector equations,

$$\underline{r} = \underline{r_1} + t\underline{a} \ \text{ and } \ \underline{r} = \underline{r_2} + t\underline{a}.$$



In the diagram, F is the foot of the perpendicular onto the second line from the point P_1 on the first line and the length of this perpendicular is d.

Hence,

$$d^2 = (\underline{r_2} - \underline{r_1}) \bullet (\underline{r_2} - \underline{r_1}) - \left[\frac{(\underline{r_2} - \underline{r_1}) \bullet \underline{a}}{a}\right]^2.$$

EXAMPLE

Determine the shortest distance between the straight line passing through the point with position vector $\underline{r_1} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, parallel to the vector $\underline{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the straight line passing through the point with position vector $\underline{r_2} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, parallel to $\underline{b}$.

Solution

From the formula,

$$d^2 = (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - \left[\frac{(-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}}\right]^2.$$

That is,

$$d^2 = (36 + 16 + 4) - \left[\frac{-6 + 4 - 2}{\sqrt{3}}\right]^2 = 56 - \frac{16}{3} = \frac{152}{3},$$

which gives

$$d \simeq 7.12$$

8.5.6 THE SHORTEST DISTANCE BETWEEN TWO SKEW STRAIGHT LINES

Two straight lines are said to be **"skew"** if they are not parallel and do not intersect each other. It may be shown that such a pair of lines will always have a common perpendicular (that is, a straight line segment which meets both, and is perpendicular to both). Its length will be the shortest distance between the two skew lines.

For the straight lines, whose vector equations are

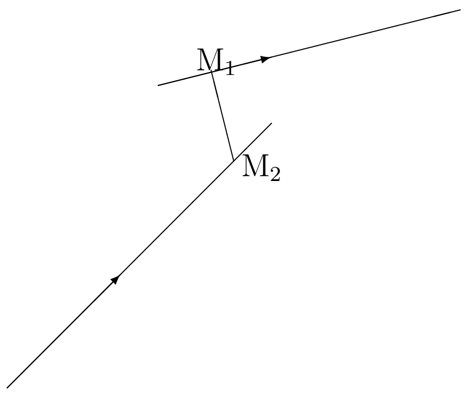
$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

suppose that the point, M_1, on the first line and the point, M_2, on the second line are the ends of the common perpendicular and have position vectors, $\underline{\mathrm{m}_1}$ and $\underline{\mathrm{m}_2}$, respectively.

Then,

$$\underline{\mathrm{m}_1} = \underline{\mathrm{r}_1} + t_1\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{m}_2} = \underline{\mathrm{r}_2} + t_2\underline{\mathrm{a}_2},$$

for some values, t_1 and t_2, of the parameter, t.



Firstly, we have

$$\underline{\mathrm{M_1M_2}} = \underline{\mathrm{m}_2} - \underline{\mathrm{m}_1} = (\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1}.$$

Secondly, a vector which is certainly perpendicular to both of the skew lines is $\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}$, so that a unit vector perpendicular to both of the skew lines is

$$\frac{\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|}.$$

This implies that

$$(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1} = \pm d\frac{\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|},$$

where d is the shortest distance between the skew lines.

Finally, if we take the scalar (dot) product of both sides of this result with the vector $\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}$, we obtain

$$(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1})\ \bullet\ (\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}) = \pm d\frac{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|^2}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|},$$

giving

$$d = \left|\ \frac{(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1})\ \bullet\ (\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2})}{|\underline{\mathrm{a}_1}\ \mathbf{x}\ \underline{\mathrm{a}_2}|}\ \right|.$$

EXAMPLE

Determine the perpendicular distance between the two skew lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = 9\mathbf{j} + 2\mathbf{k}, \quad \underline{\mathrm{a}_1} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},$$

$$\underline{r_2} = -6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}, \quad \underline{a_2} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

Solution

$$\underline{r_2} - \underline{r_1} = -6\mathbf{i} - 14\mathbf{j} + 8\mathbf{k}$$

and

$$\underline{a_1} \ \mathbf{x} \ \underline{a_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k},$$

so that

$$d = \frac{(-6)(-6) + (-14)(-15) + (8)(3)}{\sqrt{36 + 225 + 9}} = \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}.$$

8.5.7 EXERCISES

1. Determine the vector equation, and hence the parametric equations, of the straight line which passes through the point, $(5, -2, 1)$, and is parallel to the vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

2. The equations

$$\frac{-x + 2}{7} = \frac{3y - 1}{5} = \frac{2z + 1}{3}$$

determine a straight line. Determine the equation of the line in vector form, and state a set of direction ratios for this line.

3. Show that there is a point common to the two straight lines

$$\underline{r} = \underline{r_1} + t\underline{a_1} \quad \text{and} \quad \underline{r} = \underline{r_2} + t\underline{a_2},$$

where

$$\underline{r_1} = 3\mathbf{j} + 2\mathbf{k}, \quad \underline{r_2} = -2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

and

$$\underline{a_1} = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}, \quad \underline{a_2} = 9\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$$

Determine the co-ordinates of the common point.

4. Determine, in standard cartesian form, the equation of the straight line passing through the two points, $(-2, 4, 9)$ and $(2, -1, 6)$.

5. Determine the perpendicular distance of the point $(0, -2, 5)$ from the straight line which passes through the point $(1, -1, 3)$ and is parallel to the vector $3\mathbf{i} + \mathbf{j} + \mathbf{k}$.

6. Determine the shortest distance between the two parallel straight lines

$$\underline{r} = \underline{r_1} + t\underline{a} \quad \text{and} \quad \underline{r} = \underline{r_2} + t\underline{a},$$

where

$$\underline{r_1} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \underline{r_2} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

and

$$\underline{a} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

7. Determine the shortest distance between the two skew straight lines

$$\underline{r} = \underline{r_1} + t\underline{a_1} \quad \text{and} \quad \underline{r} = \underline{r_2} + t\underline{a_2},$$

where

$$\underline{r_1} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \underline{r_2} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

and

$$\underline{a_1} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \underline{a_2} = 5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

8.5.8 ANSWERS TO EXERCISES

1.

$$\underline{r} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

giving

$$x = 5 + t, \quad y = -2 - 3t, \quad z = 1 + t.$$

2. In vector form, the equation of the line is

$$\underline{\mathrm{r}} = 2\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{2}\mathbf{k} + t\left(-7\mathbf{i} + \frac{5}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}\right),$$

and set of direction ratios is

$$-42 : 10 : 9$$

3. The common point has co-ordinates $(1, 1, 1)$.

4.

$$\frac{x+2}{4} = \frac{y-4}{-5} = \frac{z-9}{-3}.$$

5.

$$d = \sqrt{\frac{62}{11}} \simeq 2.37$$

6.

$$d = \sqrt{\frac{98}{3}} \simeq 5.72$$

7.

$$d = \frac{5\sqrt{6}}{6} \simeq 2.04$$