"JUST THE MATHS"

UNIT NUMBER

8.3

VECTORS 3
(Multiplication of one vector by another)

by

A.J.Hobson

UNIT 8.3 - VECTORS 3

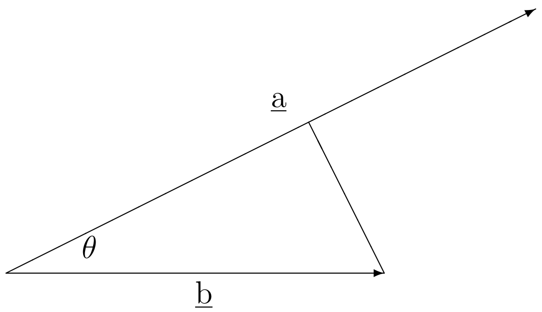
MULTIPLICATION OF ONE VECTOR BY ANOTHER

8.3.1 THE SCALAR PRODUCT (or "Dot" Product)

DEFINITION

The **"Scalar Product"** of two vectors $\underline{a}$ and $\underline{b}$ is defined as $ab\cos\theta$, where θ is the angle between the directions of $\underline{a}$ and $\underline{b}$, drawn so that they have a common end-point and are directed away from that point. The Scalar Product is denoted by $\underline{a} \bullet \underline{b}$ so that

$$\underline{a} \bullet \underline{b} = ab\cos\theta$$



$0 \leq \theta \leq \pi$

Scientific Application

If $\underline{b}$ were a force of magnitude b, then $b\cos\theta$ would be its resolution (or component) along the vector $\underline{a}$. Hence, $\underline{a} \bullet \underline{b}$ would represent the work done by $\underline{b}$ in moving an object along the vector $\underline{a}$. Similarly, if $\underline{a}$ were a force of magnitude a, then $a\cos\theta$ would be its resolution (or component) along the vector $\underline{b}$. Hence, $\underline{a} \bullet \underline{b}$ would represent the work done by $\underline{a}$ in moving an object along the vector $\underline{b}$.

8.3.2 DEDUCTIONS FROM THE DEFINITION OF DOT PRODUCT

(i) $\underline{a} \bullet \underline{a} = a^2$.

Proof:

Clearly, the angle between $\underline{a}$ and itself is zero so that

$$\underline{a} \bullet \underline{a} = a.a\cos 0 = a^2.$$

(ii) $\underline{a} \bullet \underline{b}$ can be interpreted as the magnitude of one vector times the perpendicular projection of the other vector onto it.

Proof:

$b\cos\theta$ is the perpendicular projection of $\underline{b}$ onto $\underline{a}$ and $a\cos\theta$ is the perpendicular projection of $\underline{a}$ onto $\underline{b}$.

(iii) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$.

Proof:

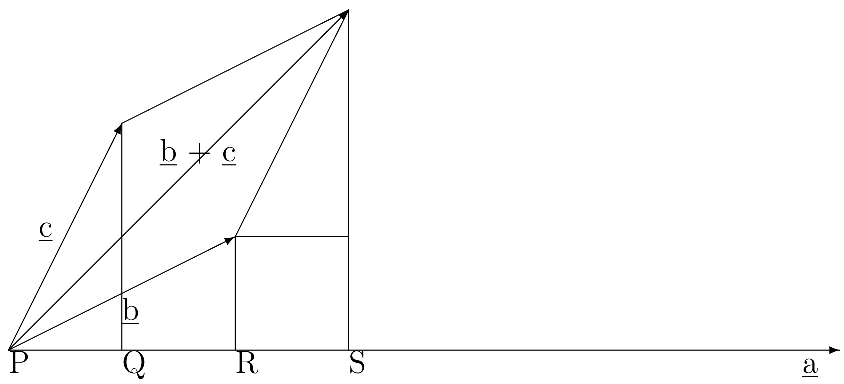
This follows since $ab\cos\theta = ba\cos\theta$.

(iv) Two non-zero vectors are perpendicular if and only if their Scalar Product is zero.

Proof:

$\underline{a}$ is perpendicular to $\underline{b}$ if and only if the angle $\theta = \frac{\pi}{2}$; that is, if and only if $\cos\theta = 0$ and hence, $ab\cos\theta = 0$.

(v) $\underline{a} \bullet (\underline{b} + \underline{c}) = \underline{a} \bullet \underline{b} + \underline{a} \bullet \underline{c}$.



The result follows from (ii) since the projections PR and PQ of $\underline{b}$ and $\underline{c}$ respectively onto $\underline{a}$ add up to the projection PS of $\underline{b} + \underline{c}$ onto $\underline{a}$.

Note:
We need to observe that RS is equal in length to PQ.

(vi) The Scalar Product of any two of the standard unit vectors **i**, **j** and **k** is given by the following multiplication table:

$\bullet$	**i**	**j**	**k**
i	1	0	0
j	0	1	0
k	0	0	1

That is, $\mathbf{i} \bullet \mathbf{i} = 1$, $\mathbf{j} \bullet \mathbf{j} = 1$ and $\mathbf{k} \bullet \mathbf{k} = 1$;

but,

$\mathbf{i} \bullet \mathbf{j} = 0$, $\mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \bullet \mathbf{k} = 0$.

8.3.3 THE STANDARD FORMULA FOR DOT PRODUCT

If

$$\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{b}} = a_1b_1 + a_2b_2 + a_3b_3.$$

Proof:

This result follows easily from the multiplication table in (vi).

Note: The angle between two vectors

If θ is the angle between the two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, then

$$\cos\theta = \frac{\underline{\mathrm{a}} \bullet \underline{\mathrm{b}}}{\mathrm{ab}}.$$

Proof:

This result is just a restatement of the original definition of a Scalar Product.

EXAMPLE

If

$$\underline{\mathrm{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = 3\mathbf{j} - 4\mathbf{k},$$

then,

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{b}} = 2 \times 0 + 2 \times 3 + (-1) \times (-4) = 10.$$

Hence,

$$\cos\theta = \frac{10}{\sqrt{2^2 + 2^2 + 1^2}\sqrt{3^2 + 4^2}} = \frac{10}{15} = \frac{2}{3}.$$

Thus,

$$\theta = 48.19^\circ \quad \text{or} \quad 0.84 \text{ radians}.$$

8.3.4 THE VECTOR PRODUCT (or "Cross" Product)

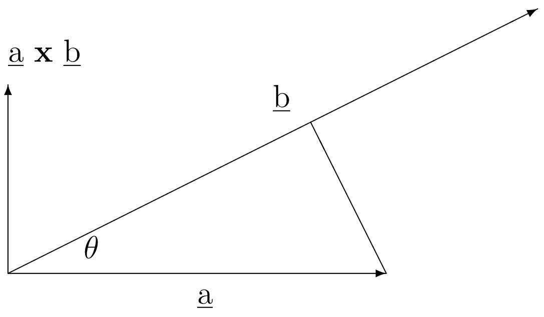
DEFINITION

If θ is the angle between two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, drawn so that they have a common end-point and are directed away from that point, then the **"Vector Product"** of $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$ is defined to be a vector of magnitude

$$\mathrm{ab}\sin\theta,$$

in a direction which is perpendicular to the plane containing $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$ and in a sense which obeys the **"right-hand-thread screw rule"** in turning from $\underline{\mathrm{a}}$ to $\underline{\mathrm{b}}$. The Vector Product is denoted by

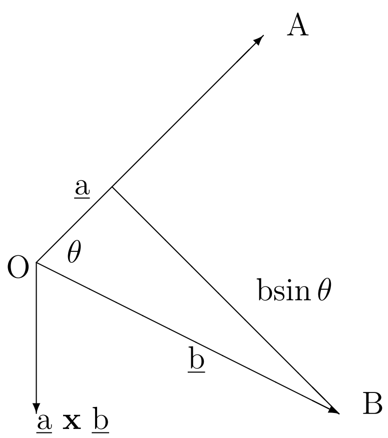
$$\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}}.$$



$$0 \leq \theta \leq \pi$$

Scientific Application

Consider the following diagram:



Suppose that the vector $\underline{\mathrm{OA}} = \underline{\mathrm{a}}$ represents a force acting at the point O and that the vector $\underline{\mathrm{OB}} = \underline{\mathrm{b}}$ is the position vector of the point B. Let the angle between the two vectors be θ.

Then the **"moment"** of the force $\underline{\mathrm{OA}}$ about the point B is a vector whose magnitude is

$$\mathrm{ab}\sin\theta$$

and whose direction is perpendicular to the plane of O, A and B in a sense which obeys the right-hand-thread screw rule in turning from $\underline{\mathrm{OA}}$ to $\underline{\mathrm{OB}}$. That is

$$\text{Moment} = \underline{\mathrm{a}}\ \ \mathbf{x}\ \ \underline{\mathrm{b}}.$$

Note:
The quantity $b \sin\theta$ is the perpendicular distance from the point B to the force $\underline{\text{OA}}$.

8.3.5 DEDUCTIONS FROM THE DEFINITION OF CROSS PRODUCT

(i)

$$\underline{a} \textbf{ x } \underline{b} = -(\underline{b} \textbf{ x } \underline{a}) = (-\underline{b}) \textbf{ x } \underline{a} = \underline{b} \textbf{ x } (-\underline{a}).$$

Proof:
This follows easily by considering the implications of the right-hand-thread screw rule.

(ii) Two vectors are parallel if and only if their Cross Product is a zero vector.

Proof:
Two vectors are parallel if and only if the angle, θ, between them is zero or π. In either case, $\sin\theta = 0$, which means that $ab\sin\theta = 0$; that is, $|\underline{a} \textbf{ x } \underline{b}| = 0$.

(iii) The Cross Product of a vector with itself is a zero vector.

Proof:
Clearly, the angle between a vector, $\underline{a}$, and itself is zero. Hence,

$$|\underline{a} \textbf{ x } \underline{a}| = a.a.\sin 0 = 0.$$

(iv)

$$\underline{a} \textbf{ x } (\underline{b} + \underline{c}) = \underline{a} \textbf{ x } \underline{b} + \underline{a} \textbf{ x } \underline{c}.$$

Proof:
This is best proved using the standard formula for a Cross Product in terms of components (see 8.3.6 below).

(v) The multiplication table for the Cross Products of the standard unit vectors **i**, **j** and **k** is as follows:

x	i	j	k
i	O	k	−j
j	− k	O	i
k	j	− i	O

That is,
i x i = O, **j x j = O**, **k x k = O**, **i x j = k**, **j x k = i**, **k x i = j**, **j x i = − k**, **k x j = − i** and **i x k = − j**.

8.3.6 THE STANDARD FORMULA FOR CROSS PRODUCT

If

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then,

$$\underline{a} \textbf{ x } \underline{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This is usually abbreviated to

$$\underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

the symbol on the right hand side being called a **"determinant"** (see Unit 7.2).

EXAMPLES

1. If $\underline{\mathrm{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{\mathrm{b}} = 3\mathbf{j} - 4\mathbf{k}$, determine $\underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{b}}$.

   Solution

$$\underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{vmatrix} = (-8+3)\mathbf{i} - (-8-0)\mathbf{j} + (6-0)\mathbf{k} = -5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}.$$

2. Show that, for any two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$,

$$(\underline{\mathrm{a}} + \underline{\mathrm{b}}) \ \mathbf{x} \ (\underline{\mathrm{a}} - \underline{\mathrm{b}}) = 2(\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{a}}).$$

   Solution

   The left hand side =

$$\underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{a}} - \underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{b}} + \underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{a}} - \underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{b}}.$$

   That is,

$$\mathbf{O} + \underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{a}} + \underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{a}} = 2(\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{a}}).$$

3. Determine the area of the triangle defined by the vectors

$$\underline{\mathrm{a}} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

   Solution

   If θ is the angle between the two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, then the area of the triangle is $\frac{1}{2}\mathrm{ab}\sin\theta$ from elementary trigonometry. The area is therefore given by

$$\frac{1}{2}|\underline{\mathrm{a}} \ \mathbf{x} \ \underline{\mathrm{b}}|.$$

   That is,

$$\text{Area} = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{matrix}\right\| = \frac{1}{2}|4\mathbf{i} + \mathbf{j} - 5\mathbf{k}|.$$

   This gives

$$\text{Area} = \frac{1}{2}\sqrt{16+1+25} = \frac{1}{2}\sqrt{42} \simeq 3.24$$

8.3.7 EXERCISES

1. In the following cases, evaluate the Scalar Product $\underline{a} \bullet \underline{b}$ and hence determine the angle, θ between $\underline{a}$ and $\underline{b}$:

   (a) $\underline{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

   (b) $\underline{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{j} + \mathbf{k}$;

   (c) $\underline{a} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\underline{b} = 7\mathbf{i} - 2\mathbf{k}$.

2. Find out which of the following pairs of vectors are perpendicular and determine the cosine of the angle between those which are not:

   (a) $3\mathbf{j}$ and $2\mathbf{j} - 2\mathbf{k}$;

   (b) $\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;

   (c) $2\mathbf{i} + 10\mathbf{k}$ and $7\mathbf{j}$;

   (d) $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

3. If $\underline{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\underline{b} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\underline{c} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, determine the length of the projection of $\underline{a} + \underline{c}$ onto $\underline{b}$.

4. If $\underline{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\underline{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, evaluate

$$(\underline{a} + \underline{b}) \bullet (\underline{a} - \underline{b}).$$

5. Determine the components of the vector $\underline{a}$ **x** $\underline{b}$ in the following cases:

   (a) $\underline{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

   (b) $\underline{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{j} + \mathbf{k}$;

   (c) $\underline{a} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\underline{b} = 7\mathbf{i} - 2\mathbf{k}$.

6. If $\underline{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\underline{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, show that $\underline{a}$ **x** $\underline{b}$ is perpendicular to the vector $\underline{c} = 9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

7. Given that $\underline{a}$ **x** $\underline{b}$ is perpendicular to each one of the vectors $\underline{a}$ and $\underline{b}$, determine a unit vector which is perpendicluar to each one of the vectors $\underline{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\underline{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
   Calculate also the sine of the angle, θ, between $\underline{a}$ and $\underline{b}$.

8. Determine the area of the triangle whose vertices are the points A(3, − 1,2), B(1, − 1, − 3) and C(4, − 3,1) in space. State your answer correct to two places of decimals.

8.3.8 ANSWERS TO EXERCISES

1. (a) Scalar Product $= -8$, $\cos\theta \simeq -0.381$ and $\theta \simeq 112.4°$;
   (b) Scalar Product $= 5$, $\cos\theta \simeq 0.645$ and $\theta \simeq 49.80°$;
   (c) Scalar Product $= -15$, $\cos\theta \simeq -0.485$ and $\theta \simeq 119.05°$
2. (a) Cosine $\simeq 0.707$;
   (b) The vectors are perpendicular;
   (c) The vectors are perpendicular;
   (d) Cosine $\simeq 0.190$
3. The length of the projection is $\frac{5}{3}$.
4. The value of the Dot Product is 24.
5. (a) The components are $-2, -7, -18$;
   (b) The components are $5, -1, 3$;
   (c) The components are $2, 26, 7$.
6. Show that $(\underline{a}\ \mathbf{x}\ \underline{b}) \bullet \underline{c} = 0$.
7. A unit vector is
$$\pm\frac{-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}}{\sqrt{155}}$$
   and
$$\sin\theta = \frac{\sqrt{155}}{\sqrt{6}.\sqrt{26}} \simeq 0.997$$
8. The area is 6.42