“JUST THE MATHS”

UNIT NUMBER

8.2

VECTORS 2
(Vectors in component form)

by

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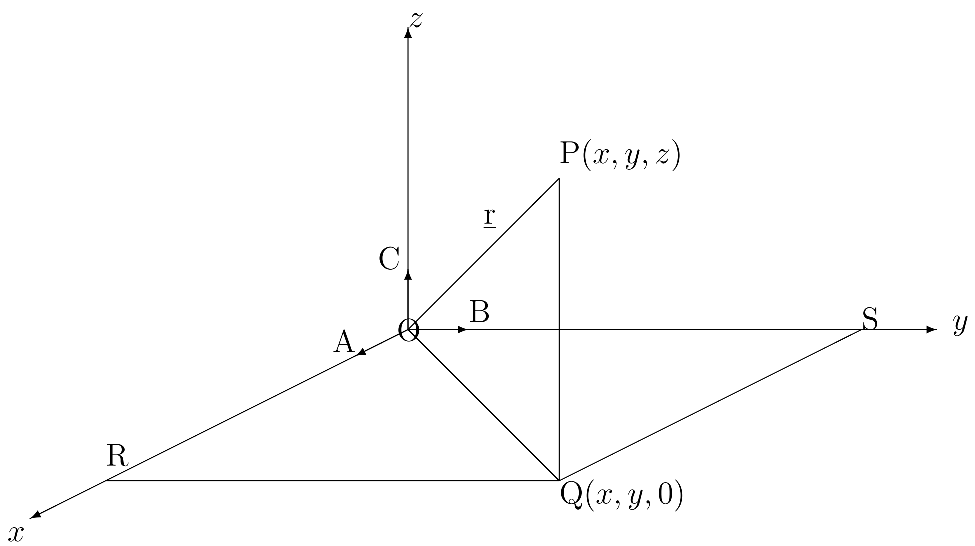
UNIT 8.2 - VECTORS 2 - VECTORS IN COMPONENT FORM

8.2.1 THE COMPONENTS OF A VECTOR

The simplest way to define a vector in space is in terms of **unit vectors** placed along the axes Ox, Oy and Oz of a three-dimensional right-handed cartesian reference system. These unit vectors will be denoted respectively by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (omitting, for convenience, the "bars" underneath and the "hats" on the top).

Consider the following diagram:



In the diagram, $\underline{\text{OA}} = \mathbf{i}$, $\underline{\text{OB}} = \mathbf{j}$ and $\underline{\text{OC}} = \mathbf{k}$. P is the point with co-ordinates (x, y, z).

By the Triangle Law

$$\underline{\text{r}} = \underline{\text{OP}} = \underline{\text{OQ}} + \underline{\text{QP}} = \underline{\text{OR}} + \underline{\text{RQ}} + \underline{\text{QP}}.$$

That is,

$$\underline{\text{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Notes:

(i) The fact that we have considered a vector which emanates from the origin is not a special case since we are dealing with free vectors. Nevertheless $\underline{\text{OP}}$ is called the position vector of the point P.

(ii) The numbers x, y and z are called the **"components"** of $\underline{\text{OP}}$ (or of any other vector in space with the same magnitude and direction as $\underline{\text{OP}}$).

(iii) To multiply (or divide) a vector in component form by a scalar, we simply multiply (or divide) each of its components by that scalar.

8.2.2 THE MAGNITUDE OF A VECTOR IN COMPONENT FORM

Referring to the diagram in section 8.2.1, Pythagoras' Theorem gives

$$(\mathrm{OP})^2 = (\mathrm{OQ})^2 + (\mathrm{QP})^2 = (\mathrm{OR})^2 + (\mathrm{RQ})^2 + (\mathrm{QP})^2.$$

That is,

$$\mathrm{r} = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}.$$

EXAMPLE

Determine the magnitude of the vector

$$\underline{\mathrm{a}} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

and hence obtain a unit vector in the same direction.

Solution

$$|\underline{\mathrm{a}}| = \mathrm{a} = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}.$$

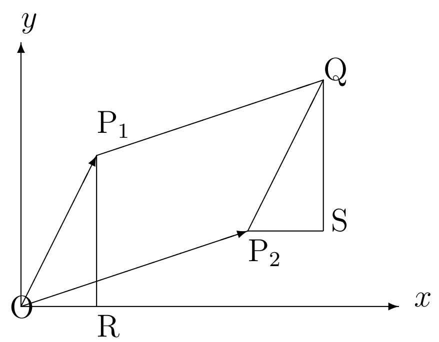
Hence, a unit vector in the same direction as $\underline{\mathrm{a}}$ is obtained by normalising $\underline{\mathrm{a}}$; that is, dividing it by its own magnitude.

The required unit vector is

$$\widehat{\underline{\mathrm{a}}} = \frac{1}{\mathrm{a}}.\underline{\mathrm{a}} = \frac{5\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{30}}.$$

8.2.3 THE SUM AND DIFFERENCE OF VECTORS IN COMPONENT FORM

We consider, first, a situation in **two** dimensions where two vectors are added together.



In the diagram, suppose P_1 has co-ordinates (x_1, y_1) and suppose P_2 has co-ordinates (x_2, y_2).

Then, since the triangle ORP_1 has exactly the same shape as the triangle P_2SQ, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

But, by the Parallelogram Law, $\underline{OQ}$ is the sum of $\underline{OP_1}$ and $\underline{OP_2}$.

That is,

$$(x_1\mathbf{i} + y_1\mathbf{j}) + (x_2\mathbf{i} + y_2\mathbf{j}) = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j},$$

showing that the sum of two vectors may be found by adding together their separate components.

It can be shown that this result applies in three dimensions also and that, to find the **difference** of two vectors, we calculate the difference of their separate components.

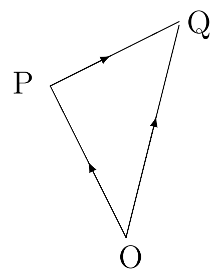
EXAMPLE

Two points P and Q in space have cartesian co-ordinates $(-3, 1, 4)$ and $(2, -2, 5)$ respectively. Determine the vector $\underline{PQ}$.

Solution

We are given that

$$\underline{OP} = -3\mathbf{i} + \mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \underline{OQ} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}.$$



By the triangle Law,

$$\underline{PQ} = \underline{OQ} - \underline{OP} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Note:
The vector $\underline{PQ}$ is, of course, the vector drawn from the point P to the point Q and it may seem puzzling that the result just obtained appears to be a vector drawn from the origin to the point $(5, -3, 1)$. However, we need to use again the fact that we are dealing with free vectors and the vector drawn from the origin to the point $(5, -3, 1)$ is parallel and equal in length to $\underline{PQ}$; in other words, it is the **same** as $\underline{PQ}$.

8.2.4 THE DIRECTION COSINES OF A VECTOR

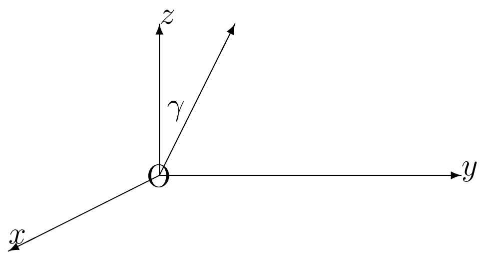
Suppose that

$$\underline{\mathrm{OP}} = \underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and suppose that $\underline{\mathrm{OP}}$ makes angles α, β and γ with Ox, Oy and Oz respectively.

Then,

$$\cos\alpha = \frac{x}{\mathrm{r}}, \quad \cos\beta = \frac{y}{\mathrm{r}} \ \text{ and } \ \cos\gamma = \frac{z}{\mathrm{r}}.$$



The three quantities $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **"direction cosines"** of $\underline{\mathrm{r}}$.

Any three numbers in the same ratio as the direction cosines are said to form a set of **"direction ratios"** for the vector $\underline{\mathrm{r}}$ and we note that $x : y : z$ is one possible set of direction ratios.

EXAMPLE

The direction cosines of the vector

$$6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

are

$$\frac{6}{\sqrt{41}}, \ \frac{2}{\sqrt{41}} \ \text{ and } \ \frac{-1}{\sqrt{41}},$$

since the vector has magnitude $\sqrt{36+4+1} = \sqrt{41}$.

A set of direction ratios for this vector are $6 : 2 : -1$.

8.2.5 EXERCISES

1. The position vectors of two points P and Q are, respectively,

$$\underline{r_1} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{r_2} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

   Determine the vector $\underline{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and hence obtain

   (a) its magnitude;

   (b) its direction cosines.

2. Obtain a unit vector which is parallel to the vector $\underline{a} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

3. If $\underline{a} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\underline{b} = -2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\underline{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, determine the following:

   (a)

$$2\underline{a} - \underline{b} + 3\underline{c};$$

   (b)

$$|\underline{a} + \underline{b} + \underline{c}|;$$

   (c)

$$|3\underline{a} - 2\underline{b} + 4\underline{c}|;$$

   (d) a unit vector which is parallel to

$$3\underline{a} - 2\underline{b} + 4\underline{c}.$$

4. Prove that the vectors $\underline{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\underline{b} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\underline{c} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ can form the sides of a triangle.

   Determine also the lengths of the **"medians"** of this triangle (that is, the lines joining each vertex to the mid-point of the opposite side).

8.2.6 ANSWERS TO EXERCISES

1. $\underline{\text{PQ}} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$.

   (a) $|\underline{\text{PQ}}| = 7$;

   (b) The direction cosines are $\frac{2}{7}$, $-\frac{6}{7}$ and $\frac{3}{7}$.

2.
$$\pm\frac{3\mathbf{i} - \mathbf{j} + 5\mathbf{k}}{\sqrt{35}}.$$

3. (a)
$$11\mathbf{i} - 8\mathbf{k};$$

   (b)
$$\sqrt{93};$$

   (c)
$$\sqrt{398};$$

   (d)
$$\pm\frac{17\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}}{\sqrt{398}}.$$

4. $\underline{a} = \underline{b} + \underline{c}$; therefore the vectors form a triangle.

   The medians have lengths equal to
$$5\sqrt{\frac{3}{2}},\ \sqrt{6} \text{ and } \frac{\sqrt{114}}{2}.$$