# "JUST THE MATHS"

# UNIT NUMBER

# 8.1

# VECTORS 1
# (Introduction to vector algebra)

## by


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## UNIT 8.1 - VECTORS 1 - INTRODUCTION TO VECTOR ALGEBRA

### 8.1.1 DEFINITIONS

1. A **"scalar"** quantity is one which has magnitude, but is not related to any direction in space.

   **Examples:** Mass, Speed, Area, Work.

2. A **"vector"** quantity is one which is specified by both a magnitude <u>and</u> a direction in space.

   **Examples:** Velocity, Weight, Acceleration.

3. A vector quantity with a fixed point of application is called a **"position vector"**.

4. A vector quantity which is restricted to a fixed <u>line</u> of action is called a **"line vector"**.

5. A vector quantity which is defined only by its magnitude and direction is called a **"free vector"**.

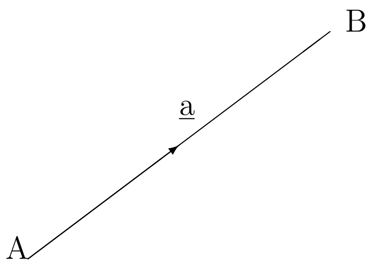
   **Note:**
   Unless otherwise stated, all vectors in the remainder of these units will be free vectors.

6. A vector quantity can be represented diagramatically by a directed straight line segment in space (with an arrow head) whose direction is that of the vector and whose length represents its magnitude according to a suitable scale.

7. The symbols $\underline{a}$, $\underline{b}$, $\underline{c}$, ...... will be used to denote vectors with magnitudes a,b,c.... but it is sometimes more convenient to use a notation such as $\underline{AB}$ which means the vector represented by the line segment drawn from the point A to the point B.

   **Notes:**
   (i) The magnitude of the vector $\underline{AB}$, which is the length of the line AB can also be denoted by the symbol $|\underline{AB}|$.

   (ii) The magnitude of the vector $\underline{a}$, which is the number a, can also be denoted by the symbol $|\underline{a}|$.




8. A vector whose magnitude is 1 is called a **"unit vector"** and the symbol $\widehat{\underline{a}}$ denotes a unit vector in the same direction as $\underline{a}$. A vector whose magnitude is zero is called a **"zero vector"** and is denoted by **O** or $\underline{O}$. It has indeterminate direction.

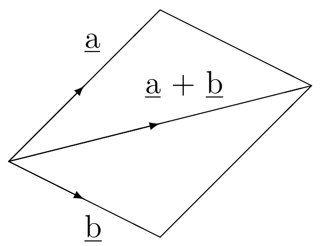
9. Two (free) vectors $\underline{a}$ and $\underline{b}$ are said to be **"equal"** if they have the same magnitude and direction.

   **Note:**
   This means that two directed straight line segments which are parallel and equal in length may be regarded as representing exactly the same vector.

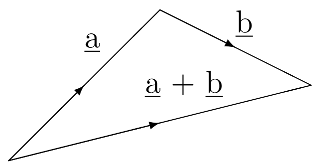
10. A vector whose magnitude is that of $\underline{a}$ but with opposite direction is denoted by $-\,\underline{a}$.

### 8.1.2 ADDITION AND SUBTRACTION OF VECTORS

Students may already know how the so-called **“resultant”** (or sum) of particular vectors, like forces, can be determined using either the **“Parallelogram Law”** or alternatively the **“Triangle Law”**. This previous knowledge is not essential here because we now define the sum of two arbitrary vectors diagramatically using either a parallelogram or a triangle. This will then lead also to a definition of subtraction for two vectors.




Parallelogram Law



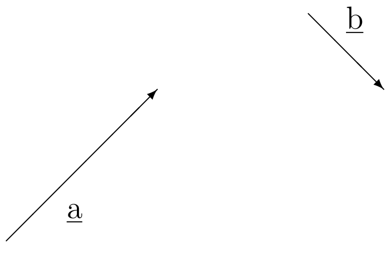

Triangle Law

**Notes:**
(i) The Triangle Law is more widely used than the Parallelogram Law because of its simplicity. We need to observe that $\underline{a}$ and $\underline{b}$ describe the triangle in the same sense while $\underline{a} + \underline{b}$ describes the triangle in the opposite sense.

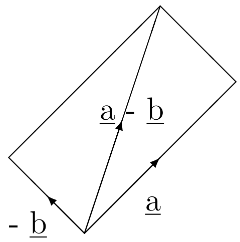
(ii) We define subtraction for vectors by considering that

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}).$$

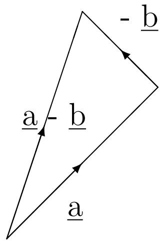
For example, to find $\underline{a} - \underline{b}$ for the vectors $\underline{a}$ and $\underline{b}$ below,



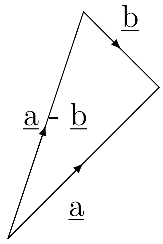

we may construct the following diagrams:




OR




OR




The third figure shows that, to find $\underline{a} - \underline{b}$, we require that $\underline{a}$ and $\underline{b}$ describe the triangle in <u>opposite</u> senses while $\underline{a} - \underline{b}$ describes the triangle in the same sense as $\underline{b}$

(iii) The sum of the three vectors describing the sides of a triangle in the same sense is always the zero vector.

### 8.1.3 MULTIPLICATION OF A VECTOR BY A SCALAR

If $m$ is any positive real number, $m\underline{a}$ is defined to be a vector in the same direction as $\underline{a}$, but of $m$ times its magnitude.

Similarly $-m\underline{a}$ is a vector in the <u>opposite</u> direction to $\underline{a}$, but of $m$ times its magnitude.

**Note:**

$\underline{a} = \mathrm{a}\hat{\underline{a}}$ and hence

$$\frac{1}{\mathrm{a}}.\underline{a} = \hat{\underline{a}}.$$

That is, if any vector is multiplied by the reciprocal of its magnitude, we obtain a unit vector in the same direction. This process is called **"normalising the vector"**.

### 8.1.4 LAWS OF ALGEBRA OBEYED BY VECTORS

**(i) The Commutative Law of Addition**

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}.$$

**(ii) The Associative Law of Addition**

$$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c} = \underline{a} + \underline{b} + \underline{c}.$$

**(iii) The Associative Law of Multiplication by a Scalar**

$$m(n\underline{a}) = (mn)\underline{a} = mn\underline{a}.$$

**(iv) The Distributive Laws for Multiplication by a Scalar**

$$(m+n)\underline{a} = m\underline{a} + n\underline{a}$$

and

$$m(\underline{a} + \underline{b}) = m\underline{a} + m\underline{b}.$$

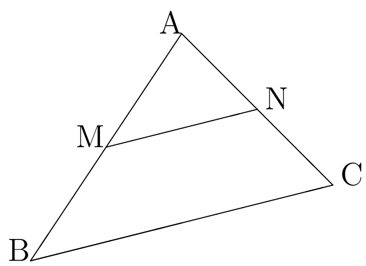
### 8.1.5 VECTOR PROOFS OF GEOMETRICAL RESULTS

The following examples illustrate how certain geometrical results which could be very cumbersome to prove using traditional geometrical methods can be much more easily proved using a vector method.

### EXAMPLES

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of its length.

   **Solution**




By the Triangle Law,

$$\underline{BC} = \underline{BA} + \underline{AC}$$

and

$$\underline{MN} = \underline{MA} + \underline{AN} = \frac{1}{2}\underline{BA} + \frac{1}{2}\underline{AC}.$$

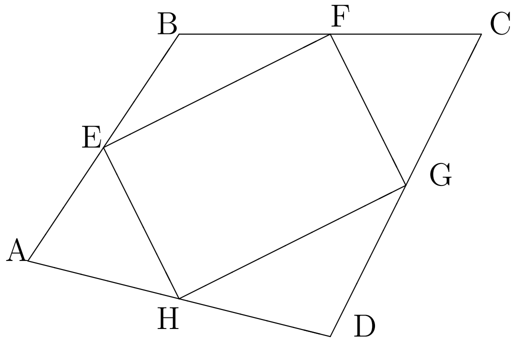
Hence,

$$\underline{MN} = \frac{1}{2}(\underline{BA} + \underline{AC}) = \frac{1}{2}\underline{BC},$$

which proves the result.

2. ABCD is a quadrilateral (four-sided figure) and E,F,G,H are the midpoints of AB, BC, CD and DA respectively. Show that EFGH is a parallelogram.

**Solution**




By the Triangle Law,

$$\underline{\text{EF}} = \underline{\text{EB}} + \underline{\text{BF}} = \frac{1}{2}\underline{\text{AB}} + \frac{1}{2}\underline{\text{BC}} = \frac{1}{2}(\underline{\text{AB}} + \underline{\text{BC}}) = \frac{1}{2}\underline{\text{AC}}$$

and also

$$\underline{\text{HG}} = \underline{\text{HD}} + \underline{\text{DG}} = \frac{1}{2}\underline{\text{AD}} + \frac{1}{2}\underline{\text{DC}} = \frac{1}{2}(\underline{\text{AD}} + \underline{\text{DC}}) = \frac{1}{2}\underline{\text{AC}}.$$

Hence,

$$\underline{\text{EF}} = \underline{\text{HG}},$$

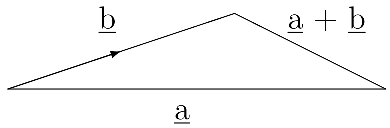
which proves the result.

## 8.1.6 EXERCISES

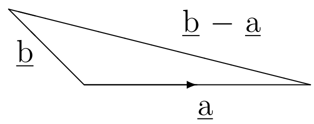
1. Which of the following are vectors and which are scalars ?
   (a) Kinetic Energy; (b) Volume; (c) Force;
   (d) Temperature; (e) Electric Field; (f) Thrust.

2. Fill in the missing arrows for the following vector diagrams:

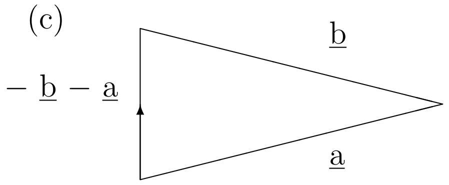
(a)



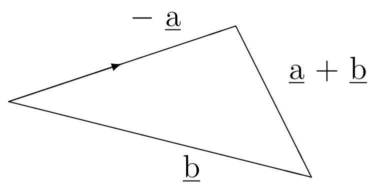

(b)




(c)




(d)




3. ABCDE is a regular pentagon with centre O. Use the Triangle Law of Addition to show that

$$\underline{AB} + \underline{BC} + \underline{CD} + \underline{DE} + \underline{EA} = \mathbf{O}.$$

4. Draw to scale a diagram which illustrates the identity

$$4\underline{a} + 3(\underline{b} - \underline{a}) = \underline{a} + 3\underline{b}.$$

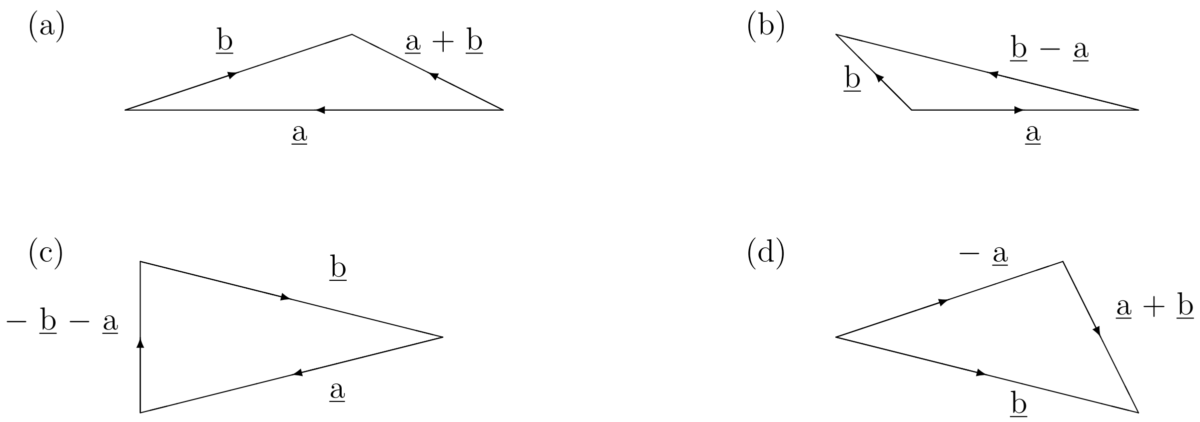
5. $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors and

$$\underline{p} = \underline{b} + \underline{c} - 2\underline{a}, \quad \underline{q} = \underline{c} + \underline{a} - 2\underline{b}, \quad \underline{r} = 3\underline{c} - 3\underline{b}.$$

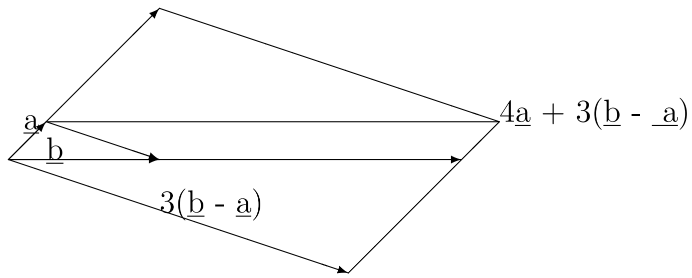
   Show that the vector $3\underline{p} - 2\underline{q}$ is parallel to the vector $5\underline{p} - 6\underline{q} + \underline{r}$.

### 8.1.7 ANSWERS TO EXERCISES

1. (a) Scalar; (b) Scalar; (c) Vector:
   (d) Scalar; (e) Vector; (f) Vector.
2. The completed diagrams are as follows:




3. Join A,B,C,D and E up to the centre, O.
4. The diagram is




5. One vector is a scalar multiple of the other.