"JUST THE MATHS"

UNIT NUMBER

7.4

DETERMINANTS 4
(Homogeneous linear equations)

by

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UNIT 7.4 - DETERMINANTS 4

HOMOGENEOUS LINEAR EQUATIONS

7.4.1 TRIVIAL AND NON-TRIVIAL SOLUTIONS

This Unit is concerned with a set of simultaneous linear equations in which all of the constant terms have value zero. Most of the discussion will involve three such **"homogeneous"** linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0. \end{aligned}$$

These could have been discussed at the same time as Cramer's Rule but are worth considering as a completely separate case since, in scientific applications, they lend themselves conveniently to the methods of row and column operations.

Observations

1. In Cramer's Rule for the above set of equations, if the determinant, Δ_0 of the coefficients of x, y and z is non-zero, there will exist a unique solution, namely $x = 0$, $y = 0$, $z = 0$, since each of the determinants Δ_1, Δ_2 and Δ_3 will contain a column of zeros (that is, the constant terms of the three equations).

   But this solution is obvious from the given set of equations and we call it the **"trivial solution"**.

2. The question arises as to whether it is possible for the set of equations to have any **"non-trivial"** solutions.

3. We shall see that non-trivial solutions occur when the number of equations reduces to less than the number of variables being solved for; that is when the equations are not linearly independent.

   For example, if one of the equations is redundant, we could solve the remaining two in an infinite number of ways by choosing one of the variables at random. Also, if two of the equations are redundant, we could solve the remaining equation in an infinite number of ways by choosing two of the variables at random.

4. It is evident from previous work that the set of homogeneous linear equations will have non-trivial solutions provided that

$$\Delta_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

5. Once it has been established that non-trivial solutions exist, it can be seen that any solution $x = \alpha$, $y = \beta$, $z = \gamma$ will imply other solutions of the form $x = \lambda\alpha$, $y = \lambda\beta$, $z = \lambda\gamma$, where λ is any non-zero number.

TYPE 1 - One of the three equations is redundant

The non-trivial solutions to a set which reduces to **two** linearly independent homogeneous linear equations in x, y and z may be stated in the form

$$x : y : z = \alpha : \beta : \gamma,$$

in which we mean that

$$\frac{x}{y} = \frac{\alpha}{\beta}, \ \frac{y}{z} = \frac{\beta}{\gamma} \text{ and } \frac{x}{z} = \frac{\alpha}{\gamma}.$$

One method of obtaining these ratios is first to eliminate z between the two equations in order to obtain the ratio $x : y$, then to eliminate y between the two equations in order to find the ratio $x : z$; but a slightly simpler method is described in the first worked example to be discussed shortly.

TYPE 2 - Two of the three equations are redundant

This case arises when the three homogeneous linear equations are multiples of one another.

Again, any solution implies an infinite number of others in the same set of ratios, $x : y : z$. But it turns out that not **all** solutions are in the same set of ratios.

For example, if the only equation remaining is

$$ax + by + cz = 0,$$

we could choose any two of the variables at random and solve for the remaining variable.

In particular, we could substitute $y = 0$ to obtain $x : y : z = -\frac{c}{a} : 0 : 1$;

and we could also substitute $z = 0$ to obtain $x : y : z = -\frac{b}{a} : 1 : 0$

From these two, it is now possible to generate solutions with any value, β, of y and any value, γ, of z (as if we had chosen y and z at random) in order to solve for x.

In fact

$$x = -(\beta.\frac{b}{a} + \gamma.\frac{c}{a}), \quad y = \beta, \quad z = \gamma.$$

Note:
It may be shown that, for a set of homogeneous linear simultaneous equations, no types of solution exist other than those discussed above.

EXAMPLES

1. Show that the homogeneous linear equations

$$\begin{aligned} 2x + y - z &= 0, \\ x - 3y + 2z &= 0, \\ x + 4y - 3z &= 0 \end{aligned}$$

have solutions other than $x = 0$, $y = 0$, $z = 0$ and determine the ratios $x : y : z$ for these non-trivial solutions.

Solution

(a)

$$\Delta_0 = \left| \begin{array}{ccc|cc} 2 & 1 & -1 & 2 & 1 \\ 1 & -3 & 2 & 1 & -3 \\ 1 & 4 & -3 & 1 & 4 \end{array} \right. = (18 + 2 - 4) - (3 + 16 - 3) = 0.$$

Thus, the equations are linearly dependent and, hence, have non-trivial solutions.

Note:
We could, alternatively, have noticed that the first equation is the sum of the second and third equations.

(b) It can always be arranged, in a set of ratios $\alpha : \beta : \gamma$, that any of the quantities which does not have to be equal to zero may be given the value 1. For example, $\frac{\alpha}{\gamma} : \frac{\beta}{\gamma} : 1$ is the same set of ratios as long as γ is not zero.

Let us now suppose that $z = 1$, giving

$$\begin{aligned} 2x + y - 1 &= 0, \\ x - 3y + 2 &= 0, \\ x + 4y - 3 &= 0 \end{aligned}$$

On solving any pair of these equations, we obtain $x = \frac{1}{7}$ and $y = \frac{5}{7}$, which means that

$$x : y : z = \frac{1}{7} : \frac{5}{7} : 1$$

That is,

$$x : y : z = 1 : 5 : 7$$

and any three numbers in these ratios form a solution.

2. Determine the values of λ for which the homogeneous linear equations

$$\begin{aligned} (1-\lambda)x + y - 2z &= 0, \\ -x + (2-\lambda)y + z &= 0, \\ y - (1-\lambda)z &= 0 \end{aligned}$$

have non-trivial solutions.

Solution

First we solve the equation

$$0 = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda) \end{vmatrix} \quad R_1 \longrightarrow R_1 - R_3$$

$$= \begin{vmatrix} 1-\lambda & 0 & -1+\lambda \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (1-\lambda)$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & -1 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \quad C_3 \longrightarrow C_3 + C_1$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix}$$

$$= -(1-\lambda)(2-\lambda)(1+\lambda)$$

Hence,

$$\lambda = 1, \quad -1 \text{ or } 2.$$

3. Determine the general solution of the homogeneous linear equation

$$3x - 7y + z = 0.$$

Solution

Substituting $y = 0$, we obtain $3x + z = 0$ and hence $x : y : z = -\frac{1}{3} : 0 : 1$.
Substituting $z = 0$, we obtain $3x - 7y = 0$ and hence $x : y : z = \frac{7}{3} : 1 : 0$
The general solution may thus be given by

$$x = \frac{7\beta}{3} - \frac{\gamma}{3}, \quad y = \beta, \quad z = \gamma$$

for arbitrary values of β and γ, though other equivalent versions are possible according to which of the three variables are chosen to have arbitrary values.

7.4.2 EXERCISES

1. Show that the homogeneous linear equations

$$\begin{aligned} x - 2y + 2z &= 0, \\ 2x - 2y - z &= 0, \\ 3x + y + z &= 0 \end{aligned}$$

   have no solutions other than the trivial solution.

2. Show that the following sets of homogeneous linear equations have non-trivial solutions and express these solutions as a set of ratios for $x : y : z$

   (a)

$$\begin{aligned} x - 2y + z &= 0, \\ x + y - 3z &= 0, \\ 3x - 3y - z &= 0; \end{aligned}$$

   (b)

$$\begin{aligned} 3x + y - 2z &= 0, \\ 2x + 4y + 2z &= 0, \\ 4x + 3y - z &= 0. \end{aligned}$$

3. Determine the values of λ for which the homogeneous linear equations

$$\begin{aligned} \lambda x + 2y + 3z &= 0, \\ 2x + (\lambda + 3)y + 6z &= 0, \\ 3x + 4y + (\lambda + 6)z &= 0 \end{aligned}$$

   have non-trivial solutions and solve them for the case when λ is an integer.

4. Determine the general solution to the homogeneous linear simultaneous equations

$$\begin{aligned} (\lambda + 1)x - 5y + 3z &= 0, \\ -2x + (\lambda - 8)y + 6z &= 0, \\ -3x - 15y + (\lambda + 11)z &= 0 \end{aligned}$$

   in the case when $\lambda = -2$.

7.4.3 ANSWERS TO EXERCISES

1.

$$\Delta_0 \neq 0.$$

2. (a)

$$x : y : z = 5 : 4 : 3;$$

(b)

$$x : y : z = 1 : -1 : 1$$

3.

$$\lambda = 1,\ 0.83 \ \text{ or } \ -10.83$$

When $\lambda = 1$, $x : y : z = -1 : -1 : 1$

4.

$$x = -5\beta + 3\gamma, \quad y = \beta, \quad z = \gamma.$$