"JUST THE MATHS"

UNIT NUMBER

7.3

DETERMINANTS 3
(Further evaluation of 3 x 3 determinants)

by

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UNIT 7.3 - DETERMINANTS 3

FURTHER EVALUATION OF THIRD ORDER DETERMINANTS

7.3.1 EXPANSION BY ANY ROW OR COLUMN

For the numerical evaluation of a third order determinant, the Rule of Sarrus is the easiest rule to apply; but we examine here some alternative versions of the original definition formula which will lead us to important standard properties of determinants.

Let us first re-state the original definition formula as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The question naturally arises as to whether the three elements of other rows (or even columns) may multiplied by their minors and the results combined in such a way as to give the same result as in the above formula. It turns out that any row or any column may be used in this way.

In order to illustrate this fact, we state again the more algebraic formula for a third order determinant, namely

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

ILLUSTRATION 1 - Expansion by the second row.

It may be observed that the expression

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

gives exactly the same result as in the original formula.

ILLUSTRATION 2 - Expansion by the third column.

It may be observed that the expression

$$c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)$$

gives exactly the same result as in the original formula.

Note:
Similar patterns of symbols give the expansions by the remaining rows and columns.

Summary
A third order determinant may be expanded (that is, evaluated) if we multiply each of the three elements in any row or (any column) by its minor then combined the results according the following pattern of so-called **"place-signs"**.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Note:
It is useful to have a special name for the **"signed-minor"** which any element of a determinant is mutiplied by, when expanding it by a row or a column. In fact every signed-minor is called a **"cofactor"**. This means that, wherever the place-sign is $+$, the minor and the cofactor are the same; but, wherever the place-sign is $-$, the cofactor is <u>numerically</u> equal to the minor but opposite in sign.

For instance,

(i) The minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$,

but the cofactor of b_1 is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

(ii) The minor and cofactor of b_2 are both equal to $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

7.3.2 ROW AND COLUMN OPERATIONS ON DETERMINANTS

INTRODUCTION

Certain types of problem in scientific work can involve determinants for which some or all of the elements are variable quantities rather than fixed numerical quantities. In these cases, the methods so far encountered for expanding a determinant may not be appropriate.

Described below is a set of standard properties for determinants of any order but, where necessary, they will be explained using either 3×3 determinants or 2×2 determinants.

STANDARD PROPERTIES OF DETERMINANTS

In this section, the methods of expanding a determinant by any row or any column will be useful to have in mind.

1. If all of the elements in a row or a column have the value zero, then the value of the determinant is equal to zero.

   Proof:

   We simply expand the determinant by the row or column of zeros.

2. If all but one of the elements in a row or column are equal to zero, then the value of the determinant is the product of the non-zero element in that row or column with its cofactor.

   Proof:

   We simply expand the determinant by the row or column containing the single non-zero element; and we also notice that the determinant is effectively equivalent to a determinant of one order lower.

   For example,

$$\begin{vmatrix} 5 & 1 & 0 \\ -2 & 4 & 3 \\ 6 & 8 & 0 \end{vmatrix} = -3\begin{vmatrix} 5 & 1 \\ 6 & 8 \end{vmatrix} = -3(40-6) = -102.$$

3. If a determinant contains two identical rows or two identical columns, then the value of the determinant is zero.

   Proof:

   If we expand the determinant by a row or column other than the two identical ones, it will turn out that all of the cofactors have value zero.

   For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

4. If two rows, or two columns, are interchanged the value of the determinant is unchanged numerically but it is reversed in sign.

   Proof:

   If we expand the determinant by a row or column other than the two which have been interchanged, then all of the cofactors will be changed in sign.

   For example,

$$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = a_1\begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - a_2\begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} + a_3\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

5. If all of the elements in a row or column have a common factor, then this common factor may be removed from the determinant and placed outside.

   Proof:

   Expanding the determinant by the row or column which contains the common factor is equivalent to removing the common factor first, then expanding by the new row or column so created.

   For example,

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

   if we expand the left-hand-side by the second column.

   Note:

   Another way of stating this property is that, if all of the elements in any row or column of a determinant are multiplied by the same factor, then the value of the determinant is also multiplied by that factor.

6. If the elements of any row in a determinant are altered by adding to them (or subtracting from them) a common multiple of the corresponding elements in another row, then the value of the determinant is unaltered. A similar result applies to columns.

   ILLUSTRATION

   The validity of this result is easily shown in the case of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = [(a_1 + kb_1)b_2 - (a_2 + kb_2)b_1] = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The above properties need not normally be used for the evaluation of determinants whose elements are simple numerical values; but, in the examples which follow, we include one such determinant in order to provide a simple introduction to the technique.

We shall use the symbols R_1, R_2 and R_3 to denote Row 1, Row 2 and Row 3; the symbols C_1, C_2 and C_3 will be used to denote Column 1, Column 2 and Column 3; and the symbol $\longrightarrow$ will stand for the word "becomes". The examples use what are called **"row operations"** and **"column operations"**.

EXAMPLES

1. Evaluate the determinant,

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix} \quad C_1 \longrightarrow C_1 \div 5;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \\ 3 & 2 & 3 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 3 & 2 & 3 \end{vmatrix} \quad R_3 \longrightarrow R_3 - 3R_1;$$

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 0 & -7 & -18 \end{vmatrix}$$

$$= 5(18 - 35) = 5 \times -17 = -85.$$

2. Solve, for x, the equation

$$\begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0.$$

Solution

We could expand the determinant directly, but we would then obtain a cubic equation in x which may not be straightforward to solve.

A better method is to try to obtain factors of this cubic equation **before** expanding the determinant.

It may be observed in this example that the three expressions in each column add up to the same quantity, namely $x + 2$. Thus if we first add Row 2 to Row 1, then add Row 3 to the new Row 1, we shall obtain $x + 2$ as a factor of the first row.

We may write

$$0 = \begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} x+2 & x+2 & x+2 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x+2)$$

$$= (x+2)\begin{vmatrix} 1 & 1 & 1 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x+2)\begin{vmatrix} 1 & 0 & 0 \\ 5 & x-4 & -4 \\ -3 & -1 & x+1 \end{vmatrix}$$

$$= (x+2)[(x-4)(x+1) - 4] = (x-2)(x^2 - 3x - 8).$$

Hence,

$$x = -2 \text{ or } x = \frac{3 \pm \sqrt{9+32}}{2} = \frac{3 \pm \sqrt{41}}{2}.$$

3. Solve, for x, the equation

$$\begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} = 0,$$

Solution

We may observe that the sum of the corresponding pairs of elements in the first two rows is the same, namely $x - 4$. Hence we may proceed as follows:

$$0 = \begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2$$

$$= \begin{vmatrix} x-4 & x-4 & x-4 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x-4)$$

$$= (x-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x-4)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x & -1 \\ 7 & 1 & x \end{vmatrix}$$

$$= (x-4)(x^2+1).$$

In this case, the only real solution is $x = 4$, the others being complex numbers $x = \pm j$.

4. Solve, for x, the equation

$$\begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix},$$

Solution

We may observe that the 2 in Row 1 may be used to reduce to zero the 4 underneath it in Row 2.

Hence,

$$0 = \begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1$$

$$= \begin{vmatrix} x & 3 & 2 \\ 4-2x & x-2 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 \div (x-2)$$

$$= (x-2)\begin{vmatrix} x & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad C_1 \longrightarrow C_1 + 2C_2$$

$$= (x-2)\begin{vmatrix} x+6 & 3 & 2 \\ 0 & 1 & 0 \\ 4 & 1 & x-1 \end{vmatrix}$$

$$= (x-2)[(x+6)(x-1)-8] = (x-2)[x^2+5x-14] = (x-2)(x+7)(x-2).$$

Thus,

$$x = 2 \ \ \text{(repeated)} \ \ \text{and} \ \ x = -7.$$

Note:
It is not possible to cover, by examples, every type of problem which may occur. The secret is first to spend a few seconds examining whether or not the sum or difference of a group of rows or columns can give a common factor immediately. If not, the procedure is to look for ways of obtaining a row or column in which all but one of the elements is zero and hence, effectively, to reduce the order of the determinant.

7.3.3 EXERCISES

1. Use row and/or column operations to evaluate the following determinants:

   (a)

$$\begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 102 & 103 & 104 \end{vmatrix};$$

   (b)

$$\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}.$$

2. Use row and/or column operations to evaluate, in terms of a and b, the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}.$$

3. Show that the equation

$$\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} = 0$$

   has one solution $x = -(a+b)$ and hence solve it completely.

4. Solve completely, for x, the following equations:

   (a)

$$\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0;$$

   (b)

$$\begin{vmatrix} x+1 & x+2 & 3 \\ 2 & x+3 & x+1 \\ x+3 & 1 & x+2 \end{vmatrix} = 0.$$

7.3.4 ANSWERS TO EXERCISES

1. (a)

$$0$$

   (b)

$$24$$

2.

$$ab$$

3.

$$x = -(a+b), \quad x = a, \quad x = b.$$

4. (a)

$$x = \frac{2}{3} \ \text{ only;}$$

   (b)

$$x = -3, \quad x = \pm\sqrt{3}.$$