"JUST THE MATHS"

UNIT NUMBER

7.2

DETERMINANTS 2
(Consistency and third order determinants)

by

A.J.Hobson

UNIT 7.2 - DETERMINANTS 2

CONSISTENCY AND THIRD ORDER DETERMINANTS

7.2.1 CONSISTENCY FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

In a genuine scientific problem involving simultaneous equations, it is not necessarily true that there will be the same number of equations to solve as there are unknowns to be determined. We examine, here, an elementary situation in which there are three equations, but only two unknowns.

Consider the set of equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0, - - - - - - - - - (2) \\ a_3x + b_3y + c_3 &= 0, - - - - - - - - - -(3) \end{aligned}$$

where we shall assume that any pair of the three equations has a unique common solution which may be calculated by Cramer's Rule.

In order that the three equations shall be consistent, the common solution of any pair must also satisfy the remaining equation. In particular, the common solution of equations (2) and (3) must also satisfy equation (1).

By Cramer's Rule in equations (2) and (3),

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

This solution will satisfy equation (1) provided that

$$a_1 \frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} - b_1 \frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + c_1 = 0.$$

In other words,

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is the determinant condition for the consistency of three simultaneous linear equations in two unknowns.

7.2.2 THE DEFINITION OF A THIRD ORDER DETERMINANT

In the consistency condition of the previous section, the expression on the left-hand-side is called a **"determinant of the third order"** and is denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

It has three **"rows"** (horizontally), three **"columns"** (vertically) and nine **"elements"** (the numbers inside the determinant).

The definition of a third order determinant may be stated in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notes:

(i) Other forms of the definition are also possible and will be encountered in Unit 7.3

(ii) The given formula for evaluating a third order determinant can be remembered by taking each element of the first row in turn and multiplying it by the so-called **"minor"** of the element, which is the second order determinant obtained by covering up the row and column in which the element appears; the results are then combined according to a $+$, $-$, $+$ pattern.

(iii) For the purpose of locating various parts of a determinant, the rows are counted from the top to the bottom and the columns are counted from the left to the right. Each row is read from the left to the right and each column is read from the top to the bottom. Thus, for example, the third element of the second column is b_3.

EXAMPLES

1. Evaluate the determinant

$$\Delta = \begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}.$$

Solution

$$\Delta = -3\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 5 & 3 \end{vmatrix} + 7\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}.$$

That is,

$$\Delta = -3(12-2) - 2(0+10) + 7(0-20) = -190.$$

2. Show that the simultaneous linear equations

$$\begin{aligned} 3x - y + 2 &= 0, \\ 2x + 5y - 1 &= 0, \\ 5x + 4y + 1 &= 0 \end{aligned}$$

are consistent (assuming that any two of the three have a common solution), and obtain the common solution.

Solution

The condition for consistency is that the determinant of coefficients and constants must be zero. We have

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & -1 \\ 5 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix}$$

$$= 3(5+4) + (2+5) + 2(8-25) = 27 + 7 - 34 = 0.$$

Thus, the equations are consistent and, to obtain their common solution, we may solve (say) the first two as follows:

$$\frac{x}{\begin{vmatrix} -1 & 2 \\ 5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}}.$$

That is,

$$\frac{x}{-9} = \frac{-y}{-7} = \frac{1}{17},$$

which gives

$$x = -\frac{9}{17} \text{ and } y = \frac{7}{17}.$$

Note:
It may be observed that the given set of simultaneous equations above is not an independent set because the third equation happens to be the sum of the other two. We say that the equations are **"linearly dependent"**; and this implies that the rows of the determinant of coefficients and constants are linearly dependent in the same way.
(In this case, Row 3 = Row 1 plus Row 2).

Furthermore, it may shown that the value of a determinant is zero if and only if its rows are linearly dependent. Hence, an alternative way of proving that a set of simultaneous linear equations is a consistent set is to show that they are linearly dependent in some way.

7.2.3 THE RULE OF SARRUS

From the given definition of a third order determinant, the complete **"expansion"** of the determinant may be given, in general, as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

But it may be observed that precisely the same terms may be obtained by first constructing a diagram which consists of the original determinant with the first two columns written out again to the right of this determinant. That is:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}.$$

Taking the sum of the possible products of the trios of numbers in the direction $\searrow$ and subtracting the sum of the possible products of the trios of numbers in the $\nearrow$ direction, we obtain the terms

$$(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_2c_3a_1 + c_3a_2b_1);$$

and it may be shown that these are exactly the same terms as those obtained by the original formula.

This **"Rule of Sarrus"** makes it possible to evaluate a third order determinant with an electronic calculator, almost without putting pen to paper, provided the calculator memory is used to store, then recall, the various products.

EXAMPLE

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = \left|\begin{array}{ccc|cc} -3 & 2 & 7 & -3 & 2 \\ 0 & 4 & -2 & 0 & 4 \\ 5 & -1 & 3 & 5 & -1 \end{array}\right.$$

$$= ([-3].4.3 + 2.[-2].5 + 7.0.[-1]) - (5.4.7 + [-1].[-2].[-3] + 3.0.2)$$

$$= (-36 - 20 + 0) - (140 - 6 + 0) = -56 - 134 = -190$$

as in a previous example.

7.2.4 CRAMER'S RULE FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

In the same way that, under certain conditions, two simultaneous linear equations may be solved by determinants of the second order, it is possible to show that, under certain conditions, three simultaneous linear equations in three unknowns may be solved by determinants of the third order.

The proof of this result will not be included here , but we state it for reference.

The simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \end{aligned}$$

have a common solution, given in symmetrical form, by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

which is called the **"Key"** to the solution and requires that $\Delta_0 \neq 0$.

Again the rule itself is known as **"Cramer's Rule"**.

EXAMPLE

Using the Rule of Sarrus, obtain the common solution of the simultaneous linear equations

$$\begin{aligned} x + 4y - z + 2 &= 0, \\ -x - y + 2z - 9 &= 0, \\ 2x + y - 3z + 15 &= 0. \end{aligned}$$

Solution

The **"Key"** is

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

where

(i)

$$\Delta_0 = \left| \begin{array}{ccc} 1 & 4 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & -3 \end{array} \right| \begin{array}{cc} 1 & 4 \\ -1 & -1 \\ 2 & 1 \end{array}.$$

Hence,

$$\Delta_0 = (3 + 16 + 1) - (2 + 2 + 12) = 20 - 16 = 4,$$

which is non-zero, and so we may continue:

(ii)

$$\Delta_1 = \left|\begin{array}{ccc|cc} 4 & -1 & 2 & 4 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 1 & -3 & 15 & 1 & -3 \end{array}\right. .$$

Hence,

$$\Delta_1 = (120 + 9 + 6) - (4 + 108 + 15) = 135 - 127 = 8.$$

(iii)

$$\Delta_2 = \left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 2 & -3 & 15 & 2 & -3 \end{array}\right. .$$

Hence,

$$\Delta_2 = (30 + 18 + 6) - (8 + 27 + 15) = 54 - 50 = 4.$$

(iv)

$$\Delta_3 = \left|\begin{array}{ccc|cc} 1 & 4 & 2 & 1 & 4 \\ -1 & -1 & -9 & -1 & -1 \\ 2 & 1 & 15 & 2 & 1 \end{array}\right. .$$

Hence,

$$\Delta_3 = (-15 - 72 - 2) - (-4 - 9 - 60) = -89 + 73 = -16.$$

(v) The solutions are therefore

$$x = -\frac{\Delta_1}{\Delta_0} = -\frac{8}{4} = -2;$$

$$y = \frac{\Delta_2}{\Delta_0} = \frac{4}{4} = 1;$$

$$z = -\frac{\Delta_3}{\Delta_0} = -\frac{-16}{4} = 4.$$

Special Cases

If it should happen that $\Delta_0 = 0$ when solving a set of three simultaneous linear equations by Cramer's Rule, earlier work has demonstrated that the rows of Δ_0 must be linearly dependent. That is the three groups of x, y and z terms must be linearly dependent.

Different situations arise according to whether or not the constant terms can also be brought in to the linear dependence relationship and we illustrate with examples as follows:

EXAMPLES

1. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 4 &= 0, \end{aligned}$$

   the third equation is the difference between the first two and hence it is redundant.

   Any solution common to the first two equations will thus be an acceptable solution. In this case, there will be an infinite number of solutions since, for example, we may choose the variable z at random, solving for x and y to obtain

$$x = \frac{11 - 5z}{5} \quad \text{and} \quad y = \frac{5z - 3}{5}.$$

2. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 7 &= 0, \end{aligned}$$

the third equation is inconsistent with the difference between the first two equations. That is,

$$x - 3y + 4z - 7 = 0 \ \text{ is } \ \text{inconsistent} \ \text{ with } \ x - 3y + 4z - 4 = 0.$$

In this case, there are no common solutions.

3. For the simultaneous linear equations

$$\begin{aligned} x - 2y + 3z - 1 &= 0, \\ 2x - 4y + 6z - 2 &= 0, \\ 3x - 6y + 9z - 3 &= 0, \end{aligned}$$

we have only one independent equation since the second and third equations are multiples of the first equation.

Again, there will be an infinite number of solutions which may be obtained by choosing two of the variables at random, then determining the corresponding value of the remaining variable.

Summary of the special cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

7.2.5 EXERCISES

1. Show that the simultaneous linear equations

$$\begin{aligned} x + y + 2 &= 0, \\ 3x + 2y - 1 &= 0, \\ 2x + y - 3 &= 0, \end{aligned}$$

are consistent and determine their common solution.

2. Show that the simultaneous linear equations

$$\begin{aligned} 7x - 2y + 1 &= 0, \\ 3x + 2y - 4 &= 0, \\ x - 6y - 9 &= 0, \end{aligned}$$

are inconsistent.

3. Obtain the values of λ for which the simultaneous linear equations

$$\begin{aligned} 3x + 5y + (\lambda - 2) &= 0, \\ 2x + y - 5 &= 0, \\ (\lambda - 1)x + 2y - 10 &= 0, \end{aligned}$$

are consistent.

4. Use Cramer's Rule to solve, for x, y and z, the following simultaneous linear equations:

$$\begin{aligned} 5x + 3y - z + 10 &= 0, \\ -2x - y + 4z - 1 &= 0, \\ -x + 2y - 7z - 17 &= 0. \end{aligned}$$

5. Show that the simultaneous linear equations

$$\begin{aligned} x - y + 7z - 1 &= 0, \\ x + 2y - 3z + 5 &= 0, \\ 5x + 4y + 5z + 13 &= 0 \end{aligned}$$

are linearly dependent and obtain the common solution for which $z = -1$.

7.2.6 ANSWERS TO EXERCISES

1.

$$x = 5 \quad y = -7.$$

2.

$$\Delta_0 \neq 0.$$

3.

$$\lambda = 5 \quad \text{or} \quad \lambda = -23.$$

4.

$$x = -4 \quad y = 3 \quad z = -1.$$

5.

$$x = \frac{8}{3} \quad y = -\frac{16}{3} \quad z = -1.$$