"JUST THE MATHS"

UNIT NUMBER

7.1

DETERMINANTS 1
(Second order determinants)

by

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UNIT 7.1 - DETERMINANTS 1

SECOND ORDER DETERMINANTS

7.1.1 PAIRS OF SIMULTANEOUS LINEAR EQUATIONS

The subject of Determinants may be introduced by considering, first, a set of two simultaneous linear equations in two unknowns. We shall take them in the form

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - -(2) \end{aligned}$$

If we subtract equation (2) $\times$ b_1 from equation (1) $\times$ b_2, we obtain

$$a_1b_2x - a_2b_1x + c_1b_2 - c_2b_1 = 0.$$

Hence,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

Similarly, if we subtract equation (2) $\times$ a_1 from equation (1) $\times$ a_2, we obtain

$$a_2b_1y - a_1b_2y + a_2c_1 - a_1c_2 = 0.$$

Hence,

$$y = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

Note:
Other arrangements of the solutions for x and y are possible, but the above arrangements

have been stated for a particular purpose which will be made clear shortly under "Observations".

The Symmetrical Form of the solution

The two separate solutions for x and y may be conveniently written in **"symmetrical"** form as follows:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

7.1.2 THE DEFINITION OF A SECOND ORDER DETERMINANT

Each of the denominators in the symmetrical form of the previous section has the same general appearance, namely the difference of the products of two pairs of numbers; and we shall rewrite each denominator in a new form using a mathematical symbol called a **"second order determinant"** and defined by the statement:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

The symbol on the left-hand-side may be called either a second order determinant or a 2×2 determinant; it has two **"rows"** (horizontally), two **"columns"** (vertically) and four **"elements"** (the numbers inside the determinant).

7.1.3 CRAMER'S RULE FOR TWO SIMULTANEOUS LINEAR EQUATIONS

The symmetrical solution to the two simultaneous linear equations may now be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

provided that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0;$

or, in an abbreviated form,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

provided that $\Delta_0 \neq 0$.

This determinant rule for solving two simultaneous linear equations is called **"Cramer's Rule"** and has equivalent forms for a larger number of equations.

Note:
The interpretation of Cramer's Rule in the case when $a_1b_2 - a_2b_1 = 0$ will be dealt with as a special case after some elementary examples:

Observations

In Cramer's Rule,

1. The determinant underneath x can be remembered by covering up the x terms in the original simultaneous equations, then using the coefficients of y and the constant terms in the pattern which they occupy on the page.
2. The determinant underneath y can be remembered by covering up the y terms in the original simultaneous equations, then using the coefficients of x and the constant terms in the pattern they occupy on the page.
3. The determinant underneath 1 can be remembered by covering up the constant terms in the original simultaneous equations, then using the coefficients of x and y in the pattern they occupy on the page.
4. The final determinant is labelled Δ_0 as a reminder to evaluate it **first**; because, if it happens to be zero, there is no point in evaluating Δ_1 and Δ_2.

EXAMPLES

1. Evaluate the determinant
$$\Delta = \begin{vmatrix} 7 & -2 \\ 4 & 5 \end{vmatrix}.$$

   Solution
$$\Delta = 7 \times 5 - 4 \times (-2) = 35 + 8 = 43$$

2. Express the value of the determinant
$$\Delta = \begin{vmatrix} -p & -q \\ p & -q \end{vmatrix}$$
   in terms of p and q.

Solution

$$\Delta = (-p) \times (-q) - p \times (-q) = p.q + p.q = 2pq.$$

3. Use Cramer's Rule to solve for x and y the simultaneous linear equations

$$\begin{aligned} 5x - 3y &= -3, \\ 2x - y &= -2. \end{aligned}$$

Solution

We may first rearrange the equations in the form

$$\begin{aligned} 5x - 3y + 3 &= 0, \\ 2x - y + 2 &= 0. \end{aligned}$$

Hence, by Cramer's Rule,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

where

$$\Delta_0 = \begin{vmatrix} 5 & -3 \\ 2 & -1 \end{vmatrix} = -5 + 6 = 1;$$

$$\Delta_1 = \begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -6 + 3 = -3;$$

$$\Delta_2 = \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} = 10 - 6 = 4.$$

Thus,

$$x = \frac{\Delta_1}{\Delta_0} = -3 \text{ and } y = -\frac{\Delta_2}{\Delta_0} = -4.$$

Special Cases

When using Cramer's Rule, the determinant Δ_0 must not have the value zero. But if it **does** have the value zero, then, the simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - -(2) \end{aligned}$$

are such that

$$a_1b_2 - a_2b_1 = 0.$$

In other words,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2},$$

which means that the x and y terms in one of the equations are proportional to the x and y terms in the other equation.

Two situations may arise which may be illustrated by the following examples:

EXAMPLES

1. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 10, \end{aligned}$$

   $\Delta_0 = 0$ but the second equation is simply a multiple of the first equation. That is, one of the equations is redundant and so there exists an **infinite number of solutions**; either of the variables may be chosen at random, with the remaining variable being expressible in terms of it.

2. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 7, \end{aligned}$$

$\Delta_0 = 0$ as before, but there is an inconsistency because, if the second equation is divided by 2, we obtain

$$3x - 2y = 3.5,$$

which is inconsistent with

$$3x - 2y = 5.$$

In this case **there are no solutions at all**.

Summary of the Special Cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

7.1.4 EXERCISES

1. Write down the values of the following determinants:

   (a) $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$; (c) $\begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix}$;

   (d) $\begin{vmatrix} x & y \\ y & x \end{vmatrix}$; (e) $\begin{vmatrix} x & x \\ y & y \end{vmatrix}$; (f) $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$.

2. Use determinants (that is, 'Cramer's Rule') to solve the following sets of simultaneous linear equations:

   (a) $\begin{matrix} 19x & + & 6y & = & 39, \\ 13x & - & 8y & = & -6. \end{matrix}$; (b) $\begin{matrix} 3x & + & 4y & + & 6 & = & 0, \\ 5x & - & 3y & - & 19 & = & 0. \end{matrix}$;

   (c) $\begin{matrix} 2x & + & 1 & = & 3y, \\ x & - & 5 & = & 7y. \end{matrix}$; (d) $\begin{matrix} 4 & - & 2y & = & x, \\ 7 & + & 3y & = & 2x. \end{matrix}$;

   (e) $\begin{matrix} 3i_1 & + & 2i_2 & = & 5, \\ i_1 & - & 3i_2 & = & 7. \end{matrix}$; (f) $\begin{matrix} 2x & - & 4z & = & 5, \\ x & - & 2z & = & 1. \end{matrix}$.

3. By expanding out all of the determinants, verify the following results:

   (a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$;

   (b) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + c_1 & b_1 \\ a_2 + c_2 & b_2 \end{vmatrix} - \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$;

   (c) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 - kb_1 & b_1 \\ a_2 - kb_2 & b_2 \end{vmatrix}$ for any number, k.

7.1.5 ANSWERS TO EXERCISES

1. The values are:
   (a) 5; (b) 5; (c) -1;
   (d) $x^2 - y^2$; (e) 0; (f) $a^2 + b^2$.

2. (a)
   $$\frac{x}{-276} = \frac{-y}{621} = \frac{1}{-230};$$
   hence, $x = 1.2$ and $y = 2.7$;
   (b)
   $$\frac{x}{-58} = \frac{-y}{-87} = \frac{1}{-29};$$
   hence, $x = 2$ and $y = -3$;
   (c)
   $$\frac{x}{22} = \frac{-y}{-11} = \frac{1}{-11};$$
   hence, $x = -2$ and $y = -1$;
   (d)
   $$\frac{x}{-26} = \frac{-y}{1} = \frac{1}{-7};$$
   hence, $x = \frac{26}{7}$ and $y = \frac{1}{7}$;
   (e)
   $$\frac{i_1}{-29} = \frac{-i_2}{-16} = \frac{1}{-11};$$
   hence, $i_1 = \frac{29}{11}$ and $i_2 = \frac{-16}{11}$;
   (f)
   $$\frac{x}{-6} = \frac{-z}{3} = \frac{1}{0},$$
   which means that Cramer's rule breaks down. In fact, the second equation gives two contradictory statements $2x - 4z = 5$ and $2x - 4z = 2$.

3. By following the instructions, the results may be verified.