

“JUST THE MATHS”

UNIT NUMBER

6.6

COMPLEX NUMBERS 6
(Complex loci)

by

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COMPLEX LOCI

6.6.1 INTRODUCTION

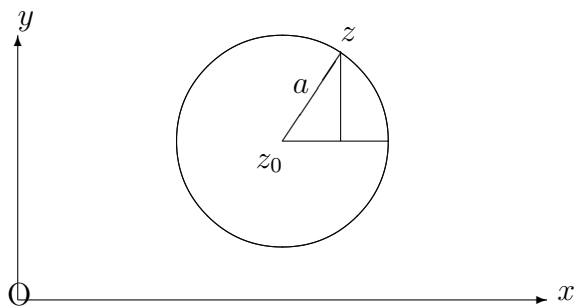
In Unit 6.2, it was mentioned that the directed line segment joining the point representing a complex number z_1 to the point representing a complex number z_2 is of length equal to $|z_2 - z_1|$ and is inclined to the positive direction of the real axis at an angle equal to $\arg(z_2 - z_1)$.

This observation now has significance when discussing variable complex numbers which are constrained to move along a certain path (or “**locus**”) in the Argand Diagram. For many practical applications, such paths (or “**loci**”) will normally be either straight lines or circles and two standard types of example appear in what follows.

In both types, we shall assume that $z = x + jy$ denotes a **variable** complex number (represented by the point (x, y) in the Argand Diagram), while $z_0 = x_0 + jy_0$ denotes a **fixed** complex number (represented by the point (x_0, y_0) in the Argand Diagram).

6.6.2 THE CIRCLE

Suppose that the moving point representing z moves on a circle, with radius a , whose centre is at the fixed point representing z_0 .



Then the distance between these two points will always be equal to a . In other words,

$$|z - z_0| = a$$

and this is the standard equation of the circle in terms of complex numbers.

Note:

By substituting $z = x + jy$ and $z_0 = x_0 + jy_0$ in the above equation, we may obtain the equivalent equation in terms of cartesian co-ordinates, namely,

$$|(x - x_0) + j(y - y_0)| = a.$$

That is,

$$(x - x_0)^2 + (y - y_0)^2 = a^2.$$

ILLUSTRATION

The equation

$$|z - 3 + j4| = 7$$

represents a circle, with radius 7, whose centre is the point representing the complex number $3 - j4$.

In cartesian co-ordinates, it is the circle with equation

$$(x - 3)^2 + (y + 4)^2 = 49.$$

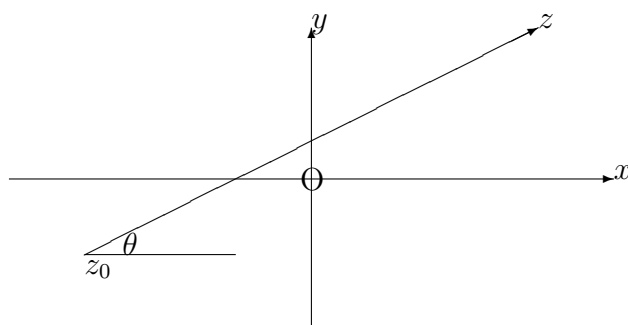
6.6.3 THE HALF-STRAIGHT-LINE

Suppose now that the “**directed**” straight line segment described **from** the fixed point representing z_0 **to** the moving point representing z is inclined at an angle θ to the positive direction of the real axis.

Then,

$$\arg(z - z_0) = \theta$$

and this equation is satisfied by **all** of the values of z for which the inclination of the directed line segment is genuinely θ and **not** $180^\circ - \theta$. The latter angle would correspond to points on the other half of the straight line joining the two points.



Note:

If we substitute $z = x + jy$ and $z_0 = x_0 + jy_0$, we obtain

$$\arg([x - x_0] + j[y - y_0]) = \theta.$$

That is,

$$\tan^{-1} \frac{y - y_0}{x - x_0} = \theta$$

or

$$y - y_0 = \tan \theta (x - x_0),$$

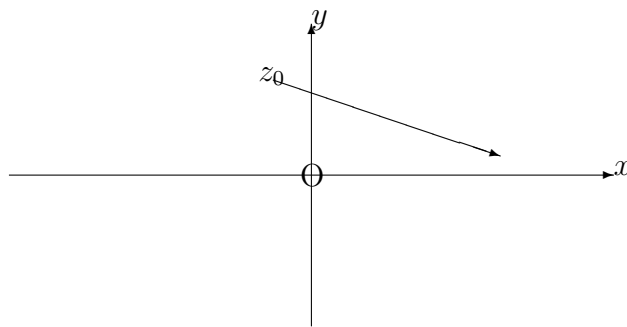
which is certainly the equation of a straight line with gradient $\tan \theta$ passing through the point (x_0, y_0) ; but it represents only that half of the straight line for which $x - x_0$ and $y - y_0$ correspond, in sign as well as value, to the real and imaginary parts of a complex number whose argument is genuinely θ and not $180^\circ - \theta$.

ILLUSTRATION

The equation

$$\arg(z + 1 - j5) = -\frac{\pi}{6}$$

represents the half-straight-line described from the point representing $z_0 = -1 + j5$ to the point representing $z = x + jy$ and inclined to the positive direction of the real axis at an angle of $-\frac{\pi}{6}$.



In terms of cartesian co-ordinates,

$$\arg([x + 1] + j[y - 5]) = -\frac{\pi}{6},$$

in which it must be true that $x + 1 > 0$ and $y - 5 < 0$ in order that the argument of $[x + 1] + j[y - 5]$ may be a negative acute angle.

We thus have the half-straight-line with equation

$$y - 5 = \tan\left(-\frac{\pi}{6}\right)(x + 1) = -\frac{1}{\sqrt{3}}(x + 1)$$

which lies to the right of, and below the point $(-1, 5)$.

6.6.4 MORE GENERAL LOCI

Certain types of locus problem may be encountered which cannot be identified with either of the two standard types discussed above. The secret, in such problems is to substitute $z = x + jy$ in order to obtain the cartesian equation of the locus. We have already seen that this method is applicable to the two standard types anyway.

ILLUSTRATIONS

1. The equation

$$\left| \frac{z-1}{z+2} \right| = \sqrt{3}$$

may be written

$$|z-1| = \sqrt{3} |z+2|.$$

That is,

$$(x-1)^2 + y^2 = 3[(x+2)^2 + y^2],$$

which simplifies to

$$2x^2 + 2y^2 + 14x + 13 = 0$$

or

$$\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{23}{4},$$

representing a circle with centre $\left(-\frac{7}{2}, 0\right)$ and radius $\sqrt{\frac{23}{4}}$.

2. The equation

$$\arg\left(\frac{z-3}{z}\right) = \frac{\pi}{4}$$

may be written

$$\arg(z-3) - \arg z = \frac{\pi}{4}.$$

That is,

$$\arg([x - 3] + jy) - \arg(x + jy) = \frac{\pi}{4}.$$

Taking tangents of both sides and using the trigonometric identity for $\tan(A - B)$, we obtain

$$\frac{\frac{y}{x-3} - \frac{y}{x}}{1 + \frac{y}{x-3} \frac{y}{x}} = 1.$$

On simplification, the equation becomes

$$x^2 + y^2 - 3x - 3y = 0$$

or

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{2},$$

the equation of a circle with centre $\left(\frac{3}{2}, \frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$.

However, we observe that the original complex number,

$$\frac{z - 3}{z},$$

cannot have an argument of $\frac{\pi}{4}$ unless its real and imaginary parts are **both** positive.

In fact,

$$\frac{z - 3}{z} = \frac{(x - 3) + jy}{x + jy} \cdot \frac{x - jy}{x - jy} = \frac{x(x - 3) + y^2 + j3}{x^2 + y^2}$$

which requires, therefore, that

$$x(x - 3) + y^2 > 0.$$

That is,

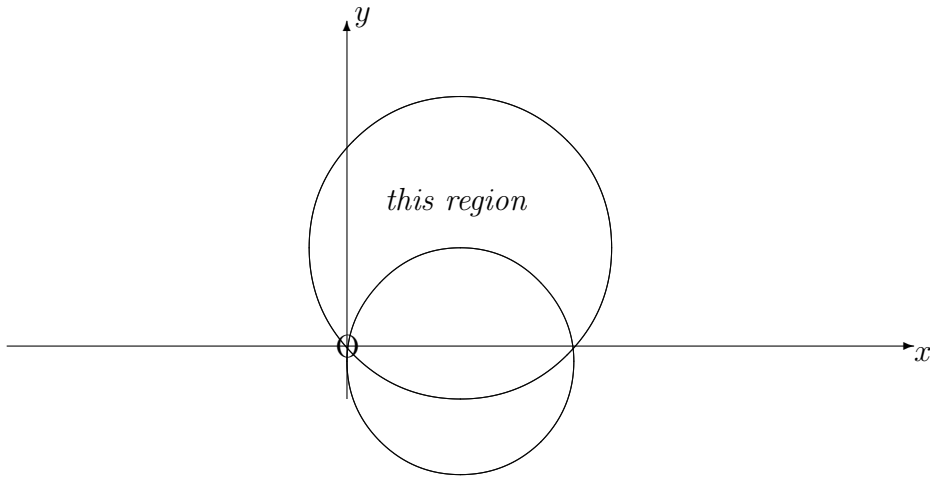
$$x^2 + y^2 - 3x > 0$$

or

$$\left(x - \frac{3}{2}\right)^2 + y^2 > \frac{9}{4}.$$

Conclusion

The locus is that part of the circle with centre $\left(\frac{3}{2}, \frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$ which lies **outside** the circle with centre $\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$.



6.6.5 EXERCISES

1. Identify the loci whose equations are

(a)

$$|z - 3| = 4;$$

(b)

$$|z - 4 + j7| = 2.$$

2. Identify the loci whose equations are

(a)

$$\arg(z + 1) = \frac{\pi}{3};$$

(b)

$$\arg(z - 2 - j3) = \frac{3\pi}{2}.$$

3. Identify the loci whose equations are

(a)

$$\left| \frac{z + j2}{z - j3} \right| = 1;$$

(b)

$$\arg\left(\frac{z + j}{z - 1}\right) = -\frac{\pi}{4}.$$

6.6.6 ANSWERS TO EXERCISES

1. (a) A circle with centre $(3, 0)$ and radius 4;
(b) A circle with centre $4, -7)$ and radius 2.
2. (a) A half-straight-line to the right of, and above the point $(-1, 0)$ inclined at an angle of $\frac{\pi}{3}$ to the positive direction of the real axis;
(b) A half-straight-line below the point $(2, 3)$ and perpendicular to the real axis.
3. (a) The straight line $y = \frac{1}{2}$;
(b) That part of the circle $x^2 + y^2 = 1$ which lies outside the circle with centre $(\frac{1}{2}, -\frac{1}{2})$ and radius $\frac{1}{\sqrt{2}}$ **and** above the straight line whose equation is $y = x - 1$.

Note:

Examples like No. 3(b) are often quite difficult and will not normally be included in the more elementary first year courses in mathematics.