"JUST THE MATHS"

UNIT NUMBER

6.5

COMPLEX NUMBERS 5
(Applications to trigonometric identities)

by

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UNIT 6.5 - COMPLEX NUMBERS 5

APPLICATIONS TO TRIGONOMETRIC IDENTITIES

6.5.1 INTRODUCTION

It will be useful for the purposes of this section to restate the result known as **"Pascal's Triangle"** previously discussed in Unit 2.2.

If n is a positive whole number, the diagram

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

provides the coefficients in the expansion of $(A+B)^n$ which contains the sequence of terms

$$A^n,\ A^{n-1}B,\ A^{n-2}B^2,\ A^{n-3}B^3, ..., B^n.$$

6.5.2 EXPRESSIONS FOR $\cos n\theta$ AND $\sin n\theta$ IN TERMS OF $\cos\theta$ AND $\sin\theta$.

From De Moivre's Theorem

$$(\cos\theta + j\sin\theta)^n \equiv \cos n\theta + j\sin n\theta,$$

from which we may deduce that, in the expansion of the left-hand-side, using Pascal's Triangle, the real part will coincide with $\cos n\theta$ and the imaginary part will coincide with $\sin n\theta$.

EXAMPLE

$$(\cos\theta + j\sin\theta)^3 \equiv \cos^3\theta + 3\cos^2\theta.(j\sin\theta) + 3\cos\theta.(j\sin\theta)^2 + (j\sin\theta)^3.$$

That is,

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta.\sin^2\theta \ \text{ or } \ 4\cos^3\theta - 3\cos\theta,$$

using $\sin^2\theta \equiv 1 - \cos^2\theta$;

and

$$\sin 3\theta \equiv 3\cos^2\theta . \sin\theta - \sin^3\theta \quad \text{or} \quad 3\sin\theta - 4\sin^3\theta,$$

using $\cos^2\theta \equiv 1 - \sin^2\theta$.

6.5.3 EXPRESSIONS FOR $\cos^n\theta$ AND $\sin^n\theta$ IN TERMS OF SINES AND COSINES OF WHOLE MULTIPLES OF θ.

The technique described here is particularly useful in calculus problems when we are required to integrate an integer power of a sine function or a cosine function. It does stand, however, as a self-contained application to trigonometry of complex numbers.

Suppose

$$z \equiv \cos\theta + j\sin\theta \quad - \quad (1)$$

Then, by De Moivre's Theorem, or by direct manipulation,

$$\frac{1}{z} \equiv \cos\theta - j\sin\theta \quad - \quad (2).$$

Adding (1) and (2) together, then subtracting (2) from (1), we obtain

$z + \frac{1}{z} \equiv 2\cos\theta$	$z - \frac{1}{z} \equiv j2\sin\theta$

Also, by De Moivre's Theorem,

$$z^n \equiv \cos n\theta + j\sin n\theta \quad - \quad (3)$$

and

$$\frac{1}{z^n} \equiv \cos n\theta - j\sin n\theta \quad - \quad (4).$$

Adding (3) and (4) together, then subtracting (4) from (3), we obtain

$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$	$z^n - \frac{1}{z^n} \equiv j2\sin n\theta$

We are now in a position to discuss some examples on finding trigonometric identities for whole number powers of $\sin\theta$ or $\cos\theta$.

EXAMPLES

1. Determine an identity for $\sin^3\theta$.

   Solution

   We use the result

$$j^3 2^3 \sin^3\theta \equiv \left(z - \frac{1}{z}\right)^3,$$

   where $z \equiv \cos\theta + j\sin\theta$.

   That is,

$$-j8\sin^3\theta \equiv z^3 - 3z^2.\frac{1}{z} + 3z.\left(\frac{1}{z}\right)^2 - \frac{1}{z^3}$$

   or, after cancelling common factors,

$$-j8\sin^3\theta \equiv z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \equiv \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right),$$

   which gives

$$-j8\sin^3\theta \equiv j2\sin 3\theta - j6\sin\theta.$$

   Hence,

$$\sin^3\theta \equiv \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right).$$

2. Determine an identity for $\cos^4\theta$.

   Solution

   We use the result

$$2^4\cos^4\theta \equiv \left(z + \frac{1}{z}\right)^4,$$

   where $z \equiv \cos\theta + j\sin\theta$.

   That is,

$$16\cos^4\theta \equiv z^4 + 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 + 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

   or, after cancelling common factors,

$$16\cos^4\theta \equiv z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

   which gives

$$16\cos^4\theta \equiv 2\cos 4\theta + 8\cos 2\theta + 6.$$

   Hence,

$$\cos^4\theta \equiv \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$$

6.5.4 EXERCISES

1. Use a complex number method to determine identities for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.
2. Use a complex number method to determine an identity for $\sin^5\theta$ in terms of sines of whole multiples of θ.
3. Use a complex number method to deternine an identity for $\cos^6\theta$ in terms of cosines of whole multiples of θ.

6.5.5 ANSWERS TO EXERCISES

1.

$$\cos 4\theta \equiv \cos^4\theta - 6\cos^2\theta.\sin^2\theta$$

and

$$\sin 4\theta \equiv 4\cos^3\theta.\sin\theta - 4\cos\theta.\sin^3\theta.$$

2.

$$\sin^5\theta \equiv \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta].$$

3.

$$\cos^6\theta \equiv \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$