"JUST THE MATHS"

UNIT NUMBER

6.4

COMPLEX NUMBERS 4
(Powers of complex numbers)

by

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UNIT 6.4 - COMPLEX NUMBERS 4

POWERS OF COMPLEX NUMBERS

6.4.1 POSITIVE WHOLE NUMBER POWERS

As an application of the rule for multiplying together complex numbers in polar form, it is a simple matter to multiply a complex number by itself any desired number of times.

Suppose that

$$z = r\angle\theta.$$

Then,

$$z^2 = r.r\angle(\theta + \theta) = r^2\angle 2\theta;$$

$$z^3 = z.z^2 = r.r^2\angle(\theta + 2\theta) = r^3\angle 3\theta;$$

and, by continuing this process,

$$z^n = r^n\angle n\theta.$$

This result is due to De Moivre, but other aspects of it will need to be discussed before we may formalise what is called **"De Moivre's Theorem"**.

EXAMPLE

$$\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)^{19} = (1\angle\left[\frac{\pi}{4}\right])^{19} = 1\angle\left[\frac{19\pi}{4}\right] = 1\angle\left[\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}.$$

6.4.2 NEGATIVE WHOLE NUMBER POWERS

If n is a negative whole number, we shall suppose that

$$n = -m,$$

where m is a positive whole number.

Thus, if $z = r\angle\theta$,

$$z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{r^m \angle m\theta}.$$

In more detail,

$$z^n = \frac{1}{r^m(\cos m\theta + j\sin m\theta)},$$

giving

$$z^n = \frac{1}{r^m}.\frac{(\cos m\theta - j\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos[-m\theta] + j\sin[-m\theta]).$$

But $-m = n$, and so

$$z^n = r^n(\cos n\theta + j\sin n\theta) = r^n \angle n\theta,$$

showing that the result of the previous section remains true for negative whole number powers.

EXAMPLE

$$(\sqrt{3} + j)^{-3} = (2\angle 30^\circ)^{-3} = \frac{1}{8}\angle(-90^\circ) = -\frac{j}{8}.$$

6.4.3 FRACTIONAL POWERS AND DE MOIVRE'S THEOREM

To begin with, here, we consider the complex number

$$z^{\frac{1}{n}},$$

where n is a positive whole number and $z = r\angle\theta$.

We define $z^{\frac{1}{n}}$ to be any complex number which gives z itself when raised to the power n. Such a complex number is called **"an n-th root of z"**.

Certainly one such possibility is

$$r^{\frac{1}{n}} \angle \frac{\theta}{n},$$

by virtue of the paragraph dealing with positive whole number powers.

But the general expression for z is given by

$$z = r\angle(\theta + k360^\circ),$$

where k may be any integer; and this suggests other possibilities for $z^{\frac{1}{n}}$, namely

$$r^{\frac{1}{n}} \angle \frac{\theta + k360^\circ}{n}.$$

However, this set of n-th roots is not an infinite set because the roots which are given by $k = 0, 1, 2, 3$...........$n - 1$ are also given by $k = n, n+1, n+2, n+3, ...,2n-1, 2n,$ $2n+1, 2n+2, 2n+3, ...$ and so on, respectively.

We conclude that there are precisely n n-th roots given by $k = 0, 1, 2, 3........., n - 1$.

EXAMPLE

Determine the cube roots (i.e. 3rd roots) of the complex number $j8$.

Solution

We first write

$$j8 = 8\angle(90^\circ + k360^\circ).$$

Hence,

$$(j8)^{\frac{1}{3}} = 8^{\frac{1}{3}} \angle \frac{(90^\circ + k360^\circ)}{3},$$

where $k = 0, 1, 2$

The three distinct cube roots are therefore

$$2\angle 30^\circ, 2\angle 150^\circ \text{ and } 2\angle 270^\circ = 2\angle(-90^\circ).$$

They all have the same modulus of 2 but their arguments are spaced around the Argand Diagram at regular intervals of $\frac{360^\circ}{3} = 120^\circ$.

Notes:

(i) In general, the n-th roots of a complex number will all have the same modulus, but their arguments will be spaced at regular intervals of $\frac{360^\circ}{n}$.

(ii) Assuming that $-180^\circ < \theta \leq 180^\circ$; that is, assuming that the polar form of z uses the principal value of the argument, then the particular n-th root of z which is given by $k = 0$ is called the **"principal n-th root"**.

(iii) If $\frac{m}{n}$ is a fraction in its lowest terms, we define

$$z^{\frac{m}{n}}$$

to be either $\left(z^{\frac{1}{n}}\right)^m$ or $(z^m)^{\frac{1}{n}}$ both of which turn out to give the same set of n distinct results.

The discussion, so far, on powers of complex numbers leads us to the following statement:

DE MOIVRE'S THEOREM

If $z = r\angle\theta$, then, for any rational number n, **one value** of z^n is $r^n\angle n\theta$.

6.4.4 EXERCISES

1. Determine the following in the form $a + jb$, expressing a and b in decimals correct to four significant figures:

   (a)
   $$(1 + j\sqrt{3})^{10};$$

   (b)
   $$(2 - j5)^{-4}.$$

2. Determine the fourth roots of $j81$ in exponential form $re^{j\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.

3. Determine the fifth roots of the complex number $-4 + j4$ in the form $a + jb$ expressing a and b in decimals, where appropriate, correct to two places. State also which root is the principal root.

4. Determine all the values of

$$(3+j4)^{\frac{3}{2}}$$

in polar form.

6.4.5 ANSWERS TO EXERCISES

1. (a)

$$(1+j\sqrt{3})^{10} = -512.0 - j886.8;$$

(b)

$$(2-j5)^{-4} = 5.796 - j1.188$$

2. The fourth roots are

$$3e^{-\frac{\pi}{8}}, \quad 3e^{\frac{3\pi}{8}}, \quad 3e^{\frac{7\pi}{8}}, \quad 3e^{-\frac{5\pi}{8}}.$$

3. The fifth roots are

$$1.26 + j0.64, \quad -0.22 + j1.40, \quad -1.40 + j0.22, \quad -0.64 - j1.26, \quad 1 - j.$$

The principal root is $1.26 + j0.64$.

4. There are two values, namely

$$11.18\angle 79.695^\circ \text{ and } 11.18\angle(-100.305^\circ).$$