"JUST THE MATHS"

UNIT NUMBER

6.3

COMPLEX NUMBERS 3
(The polar & exponential forms)

by

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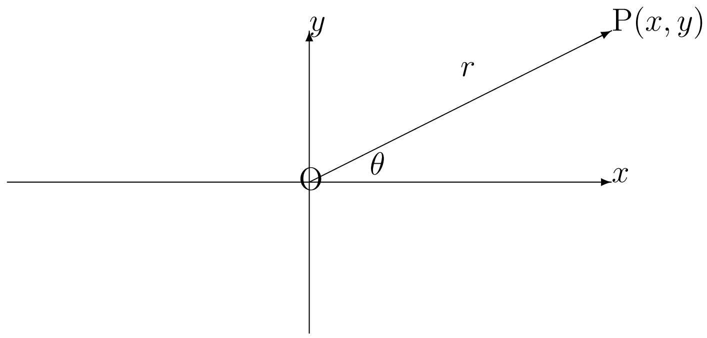
UNIT 6.3 - COMPLEX NUMBERS 3

THE POLAR AND EXPONENTIAL FORMS

6.3.1 THE POLAR FORM



From the above diagram, we may observe that

$$\frac{x}{r} = \cos\theta \ \text{ and } \ \frac{y}{r} = \sin\theta.$$

Hence, the relationship between x, y, r and θ may also be stated in the form

$$x = r\cos\theta, \quad y = r\sin\theta,$$

which means that the complex number $x + jy$ may be written as $r\cos\theta + jr\sin\theta$. In other words,

$$x + jy = r(\cos\theta + j\sin\theta).$$

The left-hand-side of this relationship is called the **"rectangular form"** or **"cartesian form"** of the complex number while the right-hand-side is called the **"polar form"**.

Note:
For convenience, the polar form may be abbreviated to $r\angle\theta$, where θ may be positive, negative or zero and may be expressed in either degrees or radians.

EXAMPLES

1. Express the complex number $z = \sqrt{3} + j$ in polar form.
   Solution
   $$|z| = r = \sqrt{3+1} = 2$$
   and
   $$\text{Arg}z = \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ + k360^\circ,$$
   where k may be any integer.
   Alternatively, using radians,
   $$\text{Arg}z = \frac{\pi}{6} + k2\pi,$$
   where k may be any integer.
   Hence, in polar form,
   $$z = 2(\cos[30^\circ + k360^\circ] + j\sin[30^\circ + k360^\circ]) = 2\angle[30^\circ + k360^\circ]$$
   or
   $$z = 2\left(\cos\left[\frac{\pi}{6} + k2\pi\right] + j\sin\left[\frac{\pi}{6} + k2\pi\right]\right) = 2\angle\left[\frac{\pi}{6} + k2\pi\right].$$

2. Express the complex number $z = -1 - j$ in polar form.
   Solution
   $$|z| = r = \sqrt{1+1} = \sqrt{2}$$
   and
   $$\text{Arg}z = \theta = \tan^{-1}(1) = -135^\circ + k360^\circ,$$
   where k may be any integer.

Alternatively,

$$\text{Arg}z = -\frac{3\pi}{4} + k2\pi,$$

where k may be any integer.

Hence, in polar form,

$$z = \sqrt{2}(\cos[-135^\circ + k360^\circ] + j\sin[-135^\circ + k360^\circ]) = \sqrt{2}\angle[-135^\circ + k360^\circ]$$

or

$$z = \sqrt{2}\left(\cos\left[-\frac{3\pi}{4} + k2\pi\right] + j\sin\left[-\frac{3\pi}{4} + k2\pi\right]\right) = \sqrt{2}\angle\left[-\frac{3\pi}{4} + k2\pi\right].$$

Note:
If it is required that the polar form should contain only the **principal** value of the argument, θ, then, provided $-180^\circ < \theta \leq 180^\circ$ or $-\pi < \theta \leq \pi$, the component $k360^\circ$ or $k2\pi$ of the result is simply omitted.

6.3.2 THE EXPONENTIAL FORM

Using some theory from the differential calculus of complex variables (not included here) it is possible to show that, for any complex number, z,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

These are, in fact, taken as the **definitions** of the functions $e^z, \sin z$ and $\cos z$.

Students who are already familiar with the differential calculus of a real variable, x, may recognise similarities between the above formulae and the "MacLaurin Series" for the functions $e^x, \sin x$ and $\cos x$. In the case of the series for $\sin x$ and $\cos x$, the value, x, must be expressed in **radians and not degrees**.

A useful deduction can be made from the three formulae if we make the substitution $z = j\theta$ into the first one, obtaining:

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

and, since $j^2 = -1$, this gives

$$e^{j\theta} = 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

On regrouping this into real and imaginary parts, then using the sine and cosine series, we obtain

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

provided θ is expressed in radians and not degrees.

The complex number $x+jy$, having modulus r and argument $\theta+k2\pi$, may thus be expressed not only in polar form but also in

the exponential form, $re^{j\theta}$.

ILLUSTRATIONS

Using the examples of the previous section

1. $$\sqrt{3} + j = 2e^{j\left(\frac{\pi}{6}+k2\pi\right)}.$$

2. $$-1 + j = \sqrt{2}e^{j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

3. $$-1 - j = \sqrt{2}e^{-j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

Note:
If it is required that the exponential form should contain only the **principal** value of the argument, θ, then, provided $-\pi < \theta \leq \pi$, the component $k2\pi$ of the result is simply omitted.

6.3.3 PRODUCTS AND QUOTIENTS IN POLAR FORM

Let us suppose that two complex numbers z_1 and z_2 have already been expressed in polar form, so that

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) = r_1\angle\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) = r_2\angle\theta_2.$$

It is then possible to establish very simple rules for determining both the product and the quotient of the two complex numbers. The explanation is as follows:

(a) The Product

$$z_1.z_2 = r_1.r_2(\cos\theta_1 + j\sin\theta_1).(\cos\theta_2 + j\sin\theta_2).$$

That is,

$$z_1.z_2 = r_1.r_2([\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$z_1.z_2 = r_1.r_2(\cos[\theta_1 + \theta_2] + j\sin[\theta_1 + \theta_2]) = r_1.r_2\angle[\theta_1 + \theta_2].$$

We have shown that, to determine the product of two complex numbers in polar form, we construct the product of their modulus values and the sum of their argument values.

(b) The Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}.$$

On multiplying the numerator and denominator by $\cos\theta_2 - j\sin\theta_2$, we obtain

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}([\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 - \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]) = \frac{r_1}{r_2}\angle[\theta_1 - \theta_2].$$

We have shown that, to determine the quotient of two complex numbers in polar form, we construct the quotient of their modulus values and the difference of their argument values.

ILLUSTRATIONS

Using results from earlier examples:

1.
$$(\sqrt{3} + j).(-1 - j) = 2\angle 30^\circ.\sqrt{2}\angle(-135^\circ) = 2\sqrt{2}\angle(-105^\circ).$$

   We notice that, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

2.
$$\frac{\sqrt{3} + j}{-1 - j} = \frac{2\angle 30^\circ}{\sqrt{2}\angle(-135^\circ)} = \sqrt{2}\angle 165^\circ.$$

   Again, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

   Note:
   It will not always turn out that the argument of a product or quotient of two complex numbers appears as the principal value. For instance,

3.
$$(-1 - j).(-\sqrt{3} - j) = \sqrt{2}\angle(-135^\circ).2\angle(-150^\circ) = 2\sqrt{2}\angle(-285^\circ),$$

   which must be converted to $2\sqrt{2}\angle(75^\circ)$ if the principal value of the argument is required.

6.3.4 EXERCISES

In the following cases, express the complex numbers z_1 and z_2 in

(a) the polar form, $r\angle\theta$

and

(b) the exponential form, $re^{j\theta}$

using only the principal value of θ.

(c) For each case, determine also the product, $z_1.z_2$, and the quotient, $\frac{z_1}{z_2}$, in polar form using only the principal value of the argument.

1.

$$z_1 = 1 + j, \quad z_2 = \sqrt{3} - j.$$

2.

$$z_1 = -\sqrt{2} + j\sqrt{2}, \quad z_2 = -3 - j4.$$

3.

$$z_1 = -4 - j5, \quad z_2 = 7 - j9.$$

6.3.5 ANSWERS TO EXERCISES

1. (a)

$$z_1 = \sqrt{2}\angle 45^\circ \quad z_2 = 2\angle(-30^\circ);$$

(b)

$$z_1 = \sqrt{2}e^{j\frac{\pi}{4}} \quad z_2 = 2e^{-j\frac{\pi}{6}};$$

(c)

$$z_1.z_2 = 2\sqrt{2}\angle 15^\circ \quad \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\angle 75^\circ.$$

2. (a)

$$z_1 = 2\angle(135^\circ) \quad z_2 = 5\angle(-127^\circ);$$

(b)

$$z_1 = 2e^{j\frac{3\pi}{4}} \quad z_2 = 5e^{-j2.22};$$

(c)

$$z_1.z_2 = 10\angle 8^\circ \quad \frac{z_1}{z_2} = \frac{2}{5}\angle(-98^\circ).$$

3. (a)

$$z_1 = 6.40\angle(-128.66^\circ) \quad z_2 = 11.40\angle(-55.13^\circ);$$

(b)

$$z_1 = 6.40e^{-j2.25} \quad z_2 = 11.40e^{-j0.96};$$

(c)

$$z_1.z_2 = 72.96\angle 176.21^\circ \quad \frac{z_1}{z_2} = 0.56\angle(-73.53^\circ).$$