"JUST THE MATHS"

UNIT NUMBER

5.9

GEOMETRY 9
(Curve sketching in general)

by

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UNIT 5.9 - GEOMETRY 9

CURVE SKETCHING IN GENERAL

Introduction

The content of the present section is concerned with those situations where it is desirable to find out the approximate shape of a curve whose equation is known, but not necessarily to determine an accurate "plot" of the curve.

In becoming accustomed to the points discussed below, the student should not feel that **every one** has to be used for a particular curve; merely enough of them to give a satisfactory impression of what the curve looks like.

5.9.1 SYMMETRY

A curve is symmetrical about the x-axis if its equation contains only even powers of y. It is symmetrical about the y-axis if its equation contains only even powers of x.

We say also that a curve is symmetrical with respect to the origin if its equation is unaltered when both x and y are changed in sign. In other words, if a point (x, y) lies on the curve, so does the point $(-x, -y)$.

ILLUSTRATIONS

1. The curve whose equation is
$$x^2\left(y^2 - 2\right) = x^4 + 4$$
is symmetrical about to both the x-axis and the y axis. This means that, once the shape of the curve is known in the first quadrant, the rest of the curve is obtained from this part by reflecting it in both axes.

   The curve is also symmetrical with respect to the origin.

2. The curve whose equation is
$$xy = 5$$
is symmetrical with respect to the origin but not about either of the co-ordinate axes.

5.9.2 INTERSECTIONS WITH THE CO-ORDINATE AXES

Any curve intersects the x-axis where $y = 0$ and the y-axis where $x = 0$; but sometimes the curve has no intersection with one or more of the co-ordinate axes. This will be borne out by an inability to solve for x when $y = 0$ or for y when $x = 0$ (or both).

ILLUSTRATION

The circle

$$x^2 + y^2 - 4x - 2y + 4 = 0$$

meets the x-axis where

$$x^2 - 4x + 4 = 0.$$

That is,

$$(x - 2)^2 = 0,$$

giving a double intersection at the point $(2, 0)$. This means that the circle **touches** the x-axis at $(2, 0)$.

The circle meets the y-axis where

$$y^2 - 2y + 4 = 0.$$

That is,

$$(y - 1)^2 = -3,$$

which is impossible, since the left hand side is bound to be positive when y is a real number.

Thus, there are no intersections with the y-axis.

5.9.3 RESTRICTIONS ON THE RANGE OF EITHER VARIABLE

It is sometimes possible to detect a range of x values or a range of y values outside of which the equation of a curve would be meaningless in terms of real geometrical points of the cartesian diagram. Usually, this involves ensuring that neither x nor y would have to assume complex number values; but other kinds of restriction can also occur.

ILLUSTRATIONS

1. The curve whose equation is
$$y^2 = 4x$$
requires that x shall not be negative; that is, $x \geq 0$.

2. The curve whose equation is
$$y^2 = x\left(x^2 - 1\right)$$
requires that the right hand side shall not be negative; and from the methods of Unit 1.10, this will be so when either $x \geq 1$ or $-1 \leq x \leq 0$.

5.9.4 THE FORM OF THE CURVE NEAR THE ORIGIN

For small values of x (or y), the higher powers of the variable can often be usefully neglected to give a rough idea of the shape of the curve near to the origin.

This method is normally applied to curves which pass **through** the origin, although the behaviour near to other points can be considered by using a temporary change of origin.

ILLUSTRATION

The curve whose equation is

$$y = 3x^3 - 2x$$

approximates to the straight line

$$y = -2x$$

for very small values of x.

5.9.5 ASYMPTOTES

DEFINITION

An **"asymptote"** is a straight line which is approached by a curve at a very great distance from the origin.

(i) Asymptotes Parallel to the Co-ordinate Axes

Consider, by way of illustration, the curve whose equation is

$$y^2 = \frac{x^3(3-2y)}{x-1}.$$

(a) By inspection, we see that the straight line $x = 1$ "meets" this curve at an infinite value of y, making it an asymptote parallel to the y-axis.

(b) Now suppose we re-write the equation as

$$x^3 = \frac{y^2(x-1)}{3-2y}.$$

Inspection, this time, suggests that the straight line $y = \frac{3}{2}$ "meets" the curve at an infinite value of x, making it an asymptote parallel to the x axis.

(c) Another way of arriving at the conclusions in (a) and (b) is to write the equation of the curve in a form without fractions, namely

$$y^2(x-1) - x^3(3-2y) = 0,$$

then equate to zero the coefficients of the highest powers of x and y. That is,

The coefficient of y^2 gives $x - 1 = 0$.

The coefficient of x^3 gives $3 - 2y = 0$.

It can be shown that this method may be used with any curve to find asymptotes parallel to the co-ordinate axes. Of course, there may not be any, in which case the method will not work.

(ii) Asymptotes in General for a Polynomial Curve

This paragraph requires a fairly advanced piece of algebraical argument, but an outline proof will be included, for the sake of completeness.

Suppose a given curve has an equation of the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y.

Then, to find the intersections with this curve of a straight line

$$y = mx + c,$$

we substitute $mx + c$ in place of y into the equation of the curve, obtaining a polynomial equation in x, say

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0.$$

In order for the line $y = mx + c$ to be an asymptote, this polynomial equation must have **coincident solutions at infinity**.

But now let us replace x by $\frac{1}{u}$ giving, after multiplying throughout by u^n, the new polynomial equation

$$a_0u^n + a_1u^{n-1} + a_2u^{n-2} + ... + a_{n-1}u + a_n = 0$$

This equation must have coincident solutions at $u = 0$ which will be the case provided

$$a_n = 0 \quad \text{and} \quad a_{n-1} = 0.$$

Conclusion

To find the asymptotes (if any) to a polynomial curve, we first substitute $y = mx + c$ into the equation of the curve. Then, in the polynomial equation obtained, we **equate to zero the two leading coefficents**; (that is, the coeffiecients of the highest two powers of x) and solve for m and c.

EXAMPLE

Determine the equations of the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution

Substituting $y = mx + c$ gives

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

That is,

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

Equating to zero the two leading coefficients; that is, the coefficients of x^2 and x, we obtain

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \quad \text{and} \quad \frac{2mc}{b^2} = 0.$$

No solution is obtainable if we let $m = 0$ in the second of these statements since it would imply $\frac{1}{a^2} = 0$ in the first statement, which is impossible. Therefore we must let $c = 0$ in the second statement, and $m = \pm\frac{b}{a}$ in the first statement.

The asymptotes are therefore

$$y = \pm\frac{b}{a}x.$$

In other words,

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 0,$$

as used earlier in the section on the hyperbola.

5.9.6 EXERCISES

1. Sketch the graphs of the following equations:

   (a)
$$y = x + \frac{1}{x};$$

   (b)
$$y = \frac{1}{x^2 + 1};$$

   (c)
$$y^2 = \frac{x}{x - 2};$$

   (d)
$$y = \frac{(x - 1)(x + 4)}{(x - 2)(x - 3)};$$

   (e)
$$y(x + 2) = (x + 3)(x - 4);$$

   (f)
$$x^2 \left(y^2 - 25\right) = y;$$

   (g)
$$y = 6 - e^{-2x}.$$

2. For each of the following curves, determine the equations of the asymptotes which are parallel to either the x-axis or the y-axis:

   (a)
$$xy^2 + x^2 - 1 = 0;$$

   (b)
$$x^2y^2 = 4\left(x^2 + y^2\right);$$

(c)

$$y = \frac{x^2 - 3x + 5}{x - 3}.$$

3. Determine all the asymptotes of the following curves:

   (a)

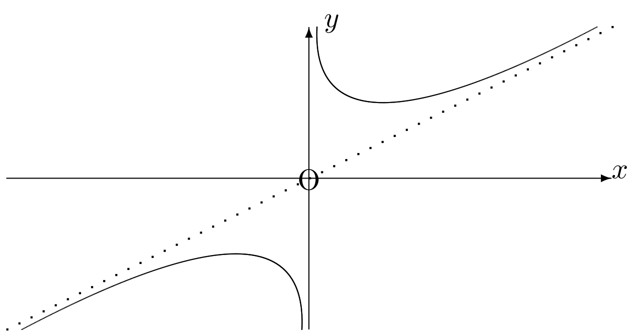
$$x^3 - xy^2 + 4x - 16 = 0;$$

   (b)

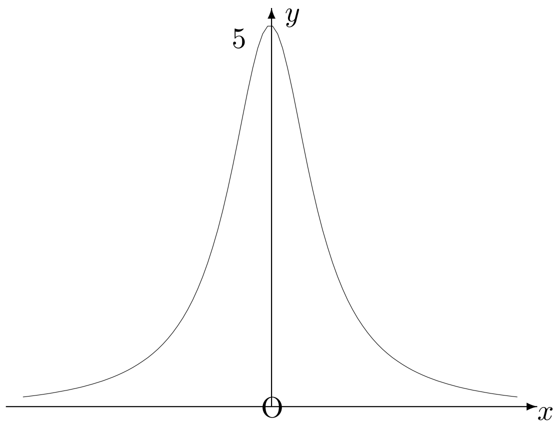
$$y^3 + 2y^2 - x^2y + y - x + 4 = 0.$$

5.9.7 ANSWERS TO EXERCISES

1.(a)



1.(b)



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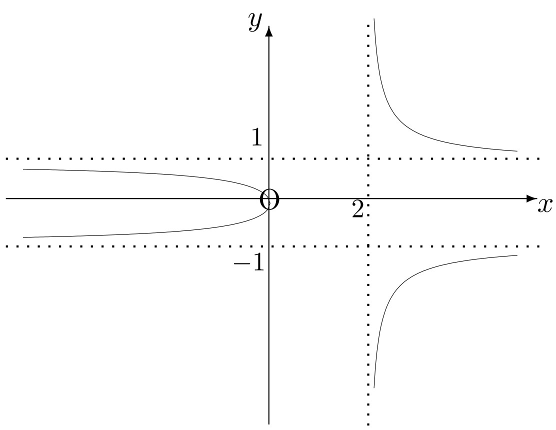

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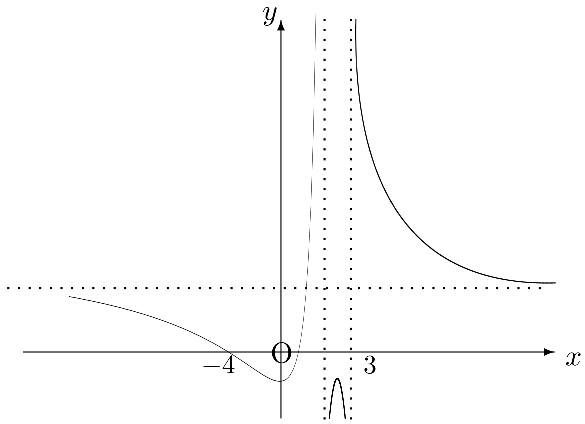

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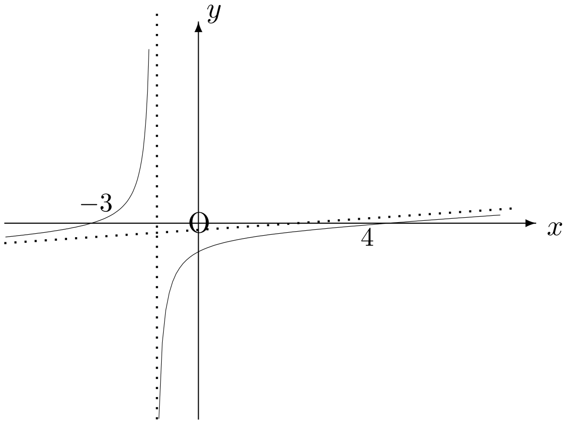

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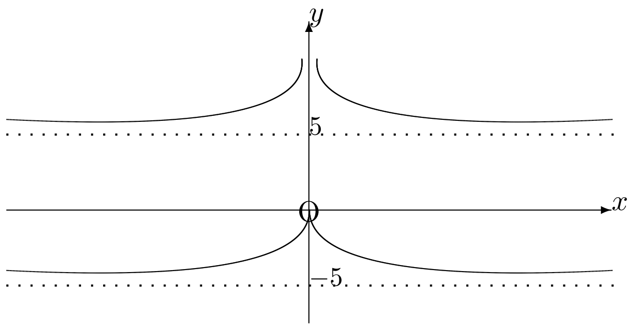
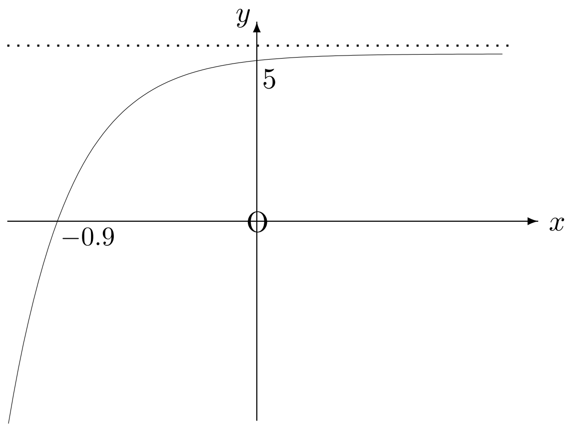


1.(g)



2.(a) $x = 0$, (b) $x = \pm 2$ and $y = \pm 2$, (c) $x = 3$

3.(a) $y = x$, $y = -x$ and $x = 0$; (b) $y = 0$, $y = x - 1$ and $y = -x - 1$.