"JUST THE MATHS"

UNIT NUMBER

5.8

GEOMETRY 8
(Conic sections - the hyperbola)

by

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UNIT 5.8 - GEOMETRY 8

CONIC SECTIONS - THE HYPERBOLA

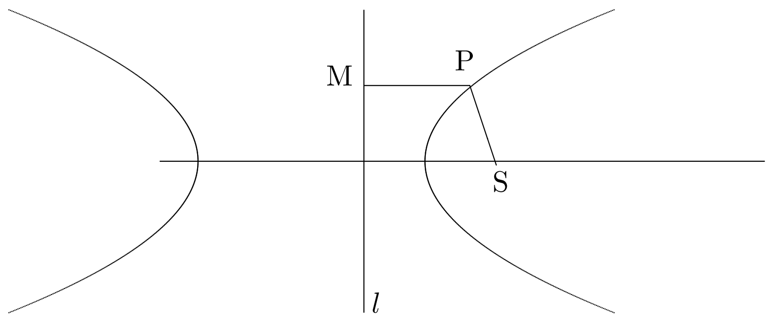
5.8.1 INTRODUCTION

DEFINITION

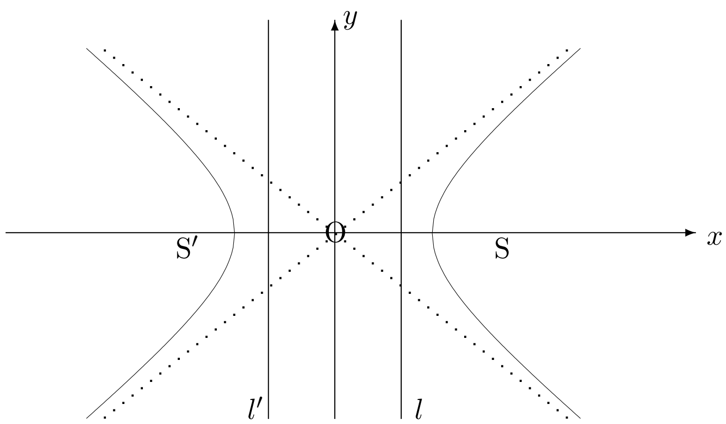
The hyperbola is the path traced out by (or **"locus"** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon > 1$ is a constant called the **"eccentricity"** of the hyperbola. The fixed line, l, is called a **"directrix"** of the hyperbola and the fixed point, S, is called a **"focus"** of the hyperbola.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci S and S′ together with two directrices l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\text{sec}\theta, \quad y = b\tan\theta$$

although, for students who meet "hyperbolic functions", a better set of parametric equations would be

$$x = a\text{cosh}t, \quad y = b\text{sinh}t.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ but does not intersect the y-axis at all.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

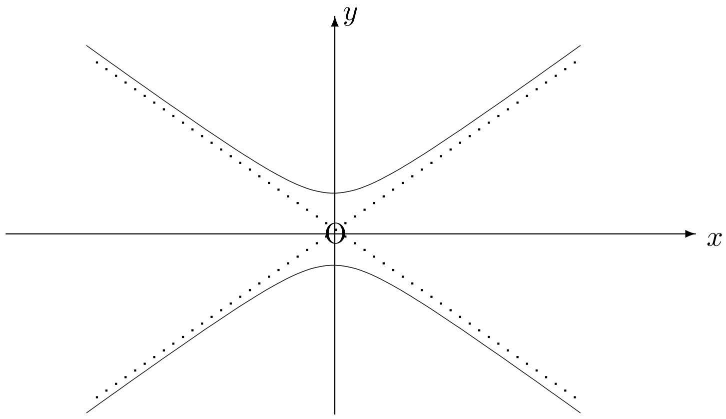
$$b^2 = a^2\left(\epsilon^2 - 1\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of a hyperbola

Note:
A similar hyperbola to the one above, but intersecting the y-axis rather than the x-axis, has equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The roles of x and y are simply reversed.



5.8.2 ASYMPTOTES

A special property of the hyperbola is that, at infinity, it approaches two straight lines through the centre of the hyperbola called **"asymptotes"**.

It can be shown that both of the hyperbolae

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

have asymptotes whose equations are:

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

These are easily obtained by factorising the **left hand side** of the equation of the hyperbola, then equating each factor to zero.

5.8.3 MORE GENERAL FORMS FOR THE EQUATION OF A HYPERBOLA

The equation of a hyperbola, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from one of the standard forms of equation by a temporary change of origin to the point (h, k). We obtain either

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

with associated parametric equations

$$x = h + a\sec\theta, \quad y = k + b\tan\theta$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

with associated parametric equations

$$x = h + a\tan\theta, \quad y = k + b\sec\theta.$$

Hyperbolae will usually be encountered in the expanded form of the above cartesian equations and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the hyperbola. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2 will be different numerically and opposite in sign.

EXAMPLE

Determine the co-ordinates of the centre and the equations of the asymptotes of the hyperbola whose equation is

$$4x^2 - y^2 + 16x + 6y + 6 = 0.$$

Solution

Completing the square in the x terms gives

$$4x^2 + 16x \equiv 4\left[x^2 + 4x\right] \equiv 4\left[(x+2)^2 - 4\right] \equiv 4(x+2)^2 - 16.$$

Completing the square in the y terms gives

$$-y^2 + 6y \equiv -\left[y^2 - 6y\right] \equiv -\left[(y-3)^2 - 9\right] \equiv -(y-3)^2 + 9.$$

Hence the equation of the hyperbola becomes

$$4(x+2)^2 - (y-3)^2 = 1$$

or

$$\frac{(x+2)^2}{\left(\frac{1}{2}\right)^2} - \frac{(y-3)^2}{1^2} = 1.$$

The centre is thus located at the point $(-2, 3)$.

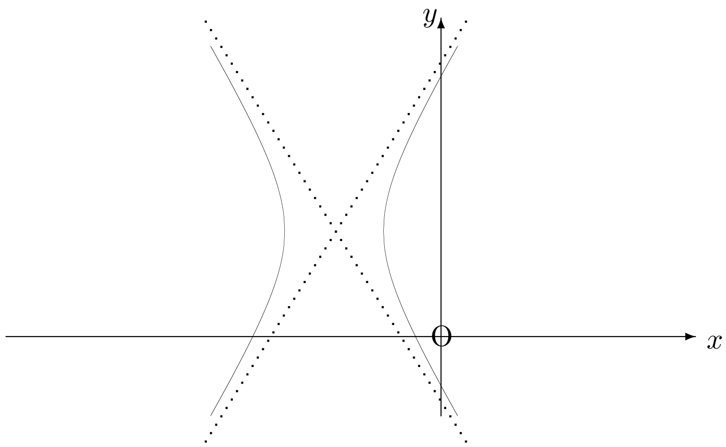
The asymptotes are best obtained by factorising the left hand side of the penultimate version of the equation of the hyperbola, then equating each factor to zero. We obtain

$$2(x+2) - (y-3) = 0 \quad \text{and} \quad 2(x+2) + (y-3) = 0.$$

In other words,

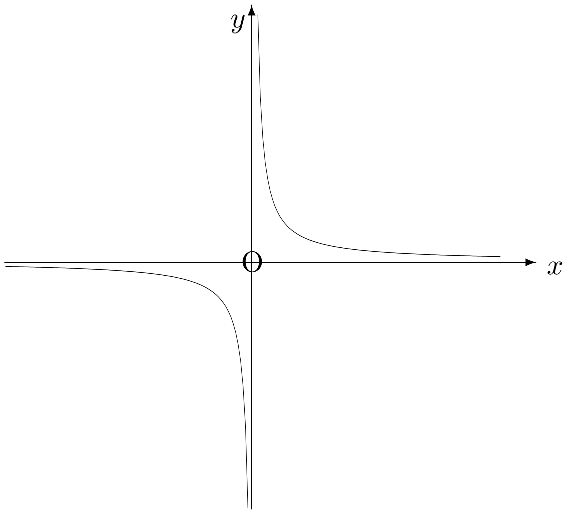
$$2x - y + 7 = 0 \quad \text{and} \quad 2x + y + 1 = 0.$$

To sketch the graph of a hyperbola, it is not always enough to have the position of the centre and the equations of the asymptotes. It may also be necessary to investigate some of the intersections of the curve with the co-ordinate axes. In the current example, by substituting first $y = 0$ and then $x = 0$ into the equation of the hyperbola, it is possible to determine intersections at $(-0.84, 0)$, $(-7.16, 0)$, $(0, -0.87)$ and $(0, 6.87)$.

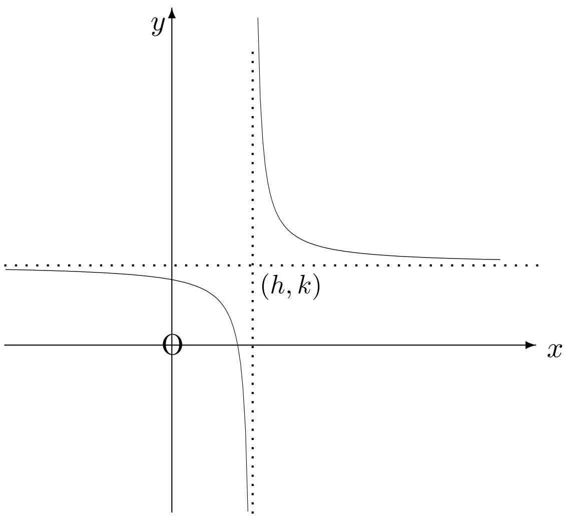


5.8.4 THE RECTANGULAR HYPERBOLA

For some hyperbolae, it will turn out that the asymptotes are at right-angles to each other; in which case, the **asymptotes themselves** could be used as the x-axis and y-axis. When this choice of reference system is made for a hyperbola with centre at the origin, it can be shown that hyperbola has the simpler equation $xy = C$, where C is a constant.



Similarly, for a rectangular hyperbola with centre at the point (h, k) and asymptotes used as the axes of reference, the equation will be $(x - h)(y - k) = C$.



Note:
A suitable pair of parametric equations for the rectangular hyperbola $(x-h)(y-k) = C$ are

$$x = t + h, \quad y = k + \frac{C}{t}.$$

EXAMPLES

1. Determine the centre of the rectangular hyperbola whose equation is

$$7x - 3y + xy - 31 = 0.$$

Solution

The equation factorises into the form

$$(x-3)(y+7) = 10.$$

Hence, the centre is located at the point $(3, -7)$.

2. A certain rectangular hyperbola has parametric equations

$$x = 1 + t, \quad y = 3 - \frac{1}{t}.$$

Determine its points of intersection with the straight line $x + y = 4$.

Solution

Substituting for x and y into the equation of the straight line, we obtain

$$1 + t + 3 - \frac{1}{t} = 4 \text{ or } t^2 - 1 = 0.$$

Hence, $t = \pm 1$ giving points of intersection at $(2, 2)$ and $(0, 4)$.

5.8.5 EXERCISES

1. For each of the following hyperbolae, determine the co-ordinates of the centre and the equations of the asympotes. Give a sketch of the curve, indicating where appropriate, the co-ordinates of its points of intersection with the x-axis and y-axis:

   (a)

   $$x^2 - y^2 - 2y = 0;$$

   (b)

   $$y^2 - x^2 - 6x = 10;$$

   (c)

   $$x^2 - y^2 - 2x - 2y = 4;$$

   (d)

   $$y^2 - x^2 - 6x + 4y = 14;$$

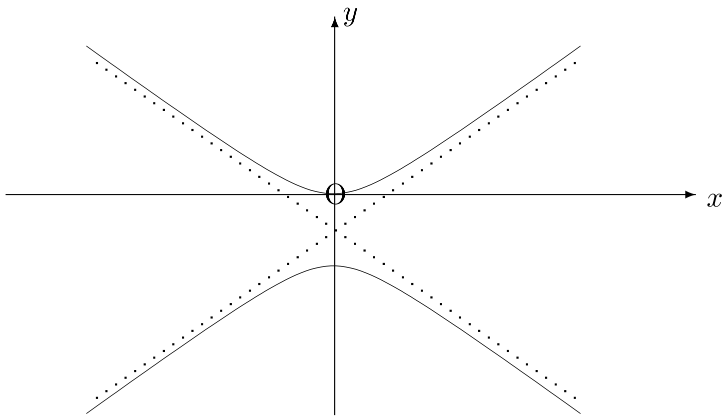
   (e)

   $$9x^2 - 4y^2 + 18x - 16y = 43.$$

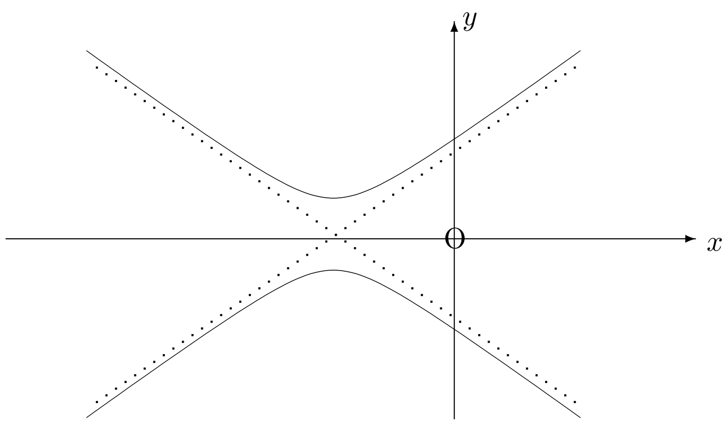
2. Determine a pair of parametric equations for the rectangular hyperbola whose equation is $xy - x + 2y - 6 = 0$ and hence obtain its points of intersection with the straight line $y = x + 3$. Sketch the hyperbola and the straight line on the same diagram.

5.8.6 ANSWERS TO EXERCISES

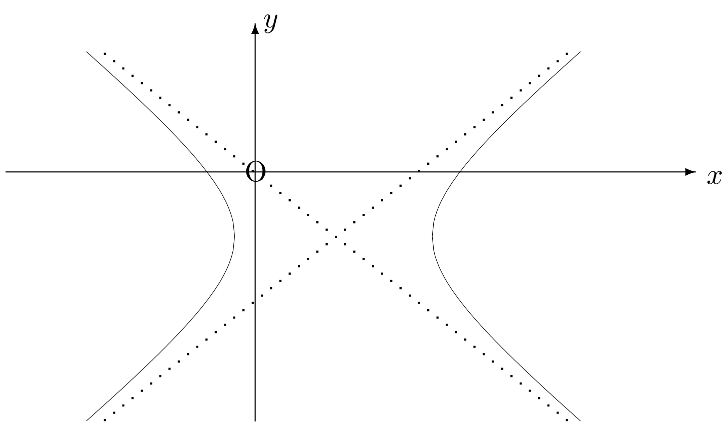
1. (a) Centre $(0, -1)$ with asymptotes $y = x - 1$ and $y = -x - 1$;



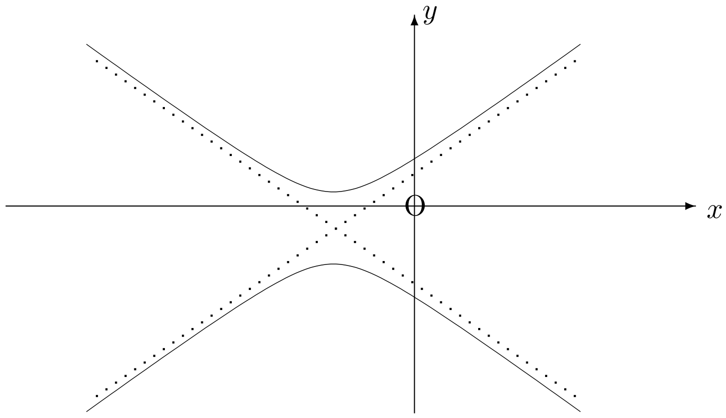
(b) Centre $(-3, 0)$ with asymptotes $y = x + 3$ and $y = -x - 3$;



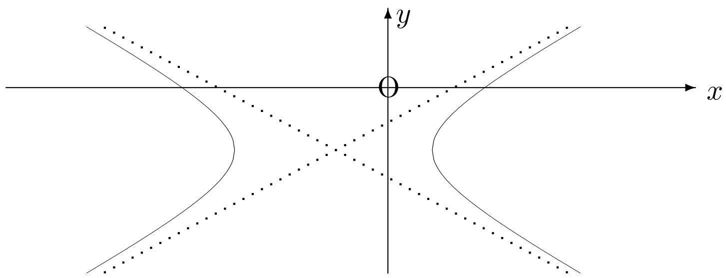
(c) Centre $(1, -1)$ with asymptotes $y = -x$ and $y = x - 2$;



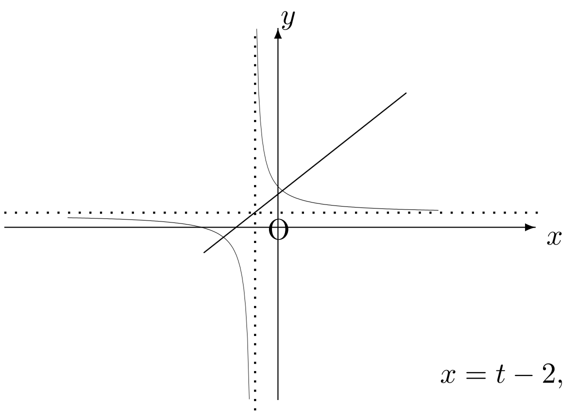
(d) Centre $(-3,-2)$ with asymptotes $y = x + 1$ and $y = -x - 5$;



(e) Centre $(-1,-2)$ with asymptotes $3x - 2y = 1$ and $3x + 2y = -7$.



2. Centre $(-2,1)$ with asymptotes $x = -2$ and $y = 1$. Intersections $(0,3)$ and $(-4,-1)$.



$$x = t - 2, \quad y = \frac{4}{t} + 1$$