"JUST THE MATHS"

UNIT NUMBER

5.7

GEOMETRY 7
(Conic sections - the ellipse)

by

A.J.Hobson

UNIT 5.7 - GEOMETRY 7

CONIC SECTIONS - THE ELLIPSE

5.7.1 INTRODUCTION

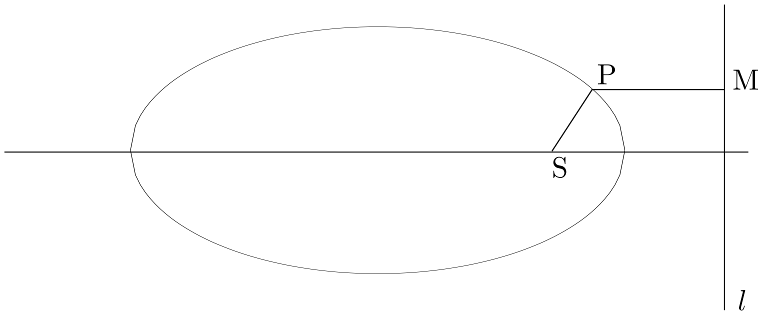
The Standard Form for the equation of an Ellipse

DEFINITION

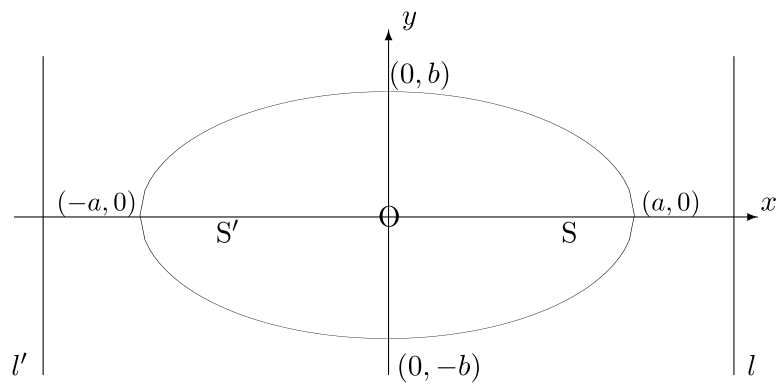
The Ellipse is the path traced out by (or **"locus** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon < 1$ is a constant called the **"eccentricity"** of the ellipse. The fixed line, l, is called a **"directrix"** of the ellipse and the fixed point, S, is called a **"focus"** of the ellipse.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci, S and S′, together with two directrices, l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\cos\theta, \quad y = b\sin\theta.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$. Whichever is the larger of a and b defines the length of the **"semi-major axis"** and whichever is the smaller defines the length of the **"semi-minor axis"**.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

$$b^2 = a^2\left(1 - \epsilon^2\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of an ellipse.

5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from the standard form of equation by a temporary change of origin to the point (h, k). We obtain

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a\cos\theta, \quad y = k + b\sin\theta.$$

Ellipses will usually be encountered in the expanded form of the above cartesian equation and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the ellipse. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2, though both of the same sign, will not be equal to each other.

EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

Solution

Completing the square in the x terms gives

$$3x^2 + 12x \equiv 3\left[x^2 + 4x\right] \equiv 3\left[(x+2)^2 - 4\right] \equiv 3(x+2)^2 - 12.$$

Completing the square in the y terms gives

$$y^2 - 2y \equiv (y-1)^2 - 1.$$

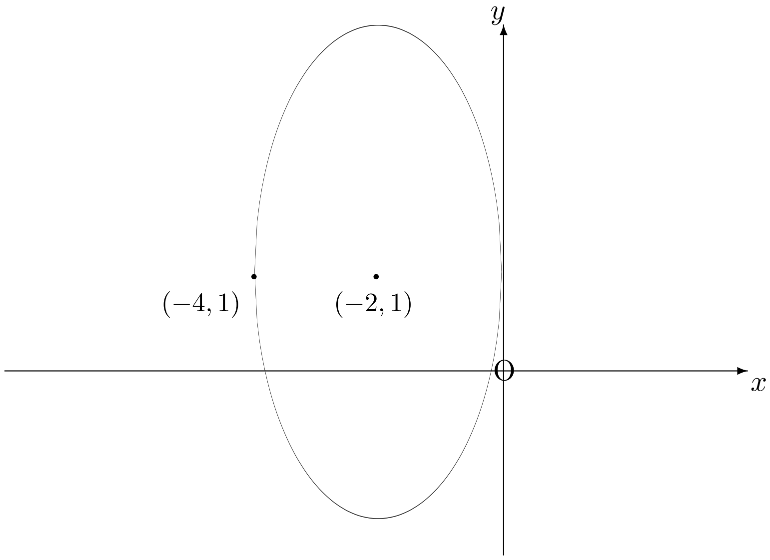
Hence, the equation of the ellipse becomes

$$3(x+2)^2 + (y-1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is thus located at the point $(-2, 1)$ and the semi-axes have lengths $a = 2$ and $b = \sqrt{12}$.



5.7.3 EXERCISES

1. For each of the following ellipses, determine the co-ordinates of the centre and give a sketch of the curve:

   (a)

   $$x^2 + 4y^2 - 4x - 8y + 4 = 0;$$

   (b)

   $$x^2 + 4y^2 + 16y + 12 = 0;$$

   (c)

   $$x^2 + 4y^2 + 6x - 8y + 9 = 0.$$

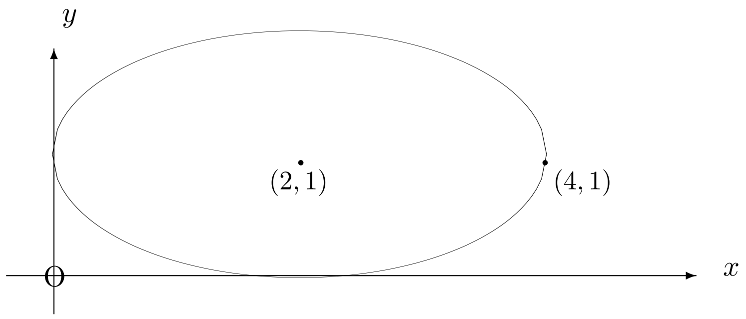
2. Determine the lengths of the semi-axes of the ellipse whose equation is

   $$9x^2 + 25y^2 = 225$$

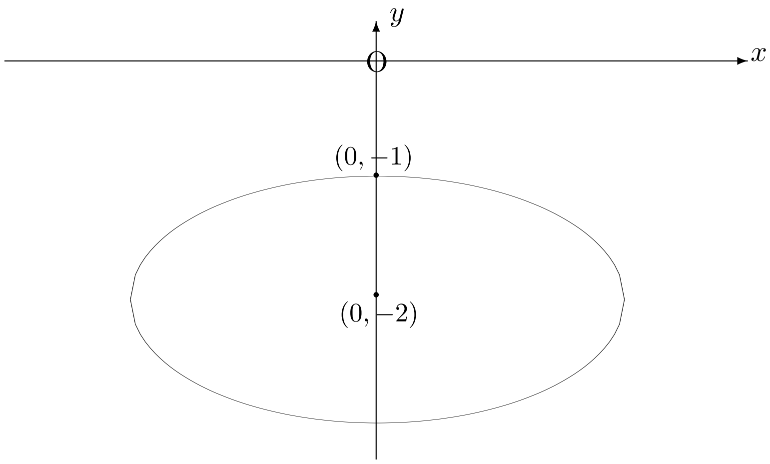
   and write down also a pair of parametric equations for this ellipse.

5.7.4 ANSWERS TO EXERCISES

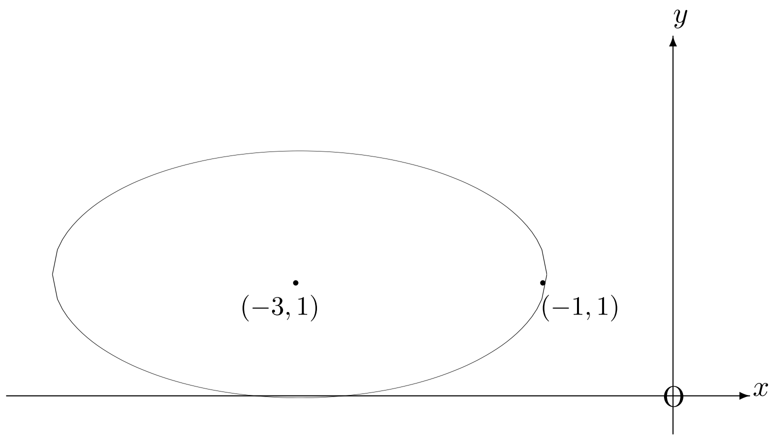
1. (a) Centre $(2, 1)$ with $a = 2$ and $b = 1$.



(b) Centre $(0, -2)$ with $a = 2$ and $b = 1$.



(c) Centre $(-3, 1)$ with $a = 2$ and $b = 1$.



2. $a = 5$ and $b = 3$, giving the parametric equations $x = 5\cos\theta, \quad y = 3\sin\theta$.