"JUST THE MATHS"

UNIT NUMBER

5.5

GEOMETRY 5
(Conic sections - the circle)

by

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UNIT 5.5 - GEOMETRY 5

CONIC SECTIONS - THE CIRCLE

5.5.1 INTRODUCTION

In this and the following three units, we shall investigate some of the geometry of four standard curves likely to be encountered in the scientific applications of Mathematics. They are the Circle, the Parabola, the Ellipse and the Hyperbola.

These curves could be generated, if desired, by considering plane sections through a cone; and, because of this, they are often called **"conic sections"** or even just **"conics"**. We shall not discuss this interpretation further, but rather use a more analytical approach.

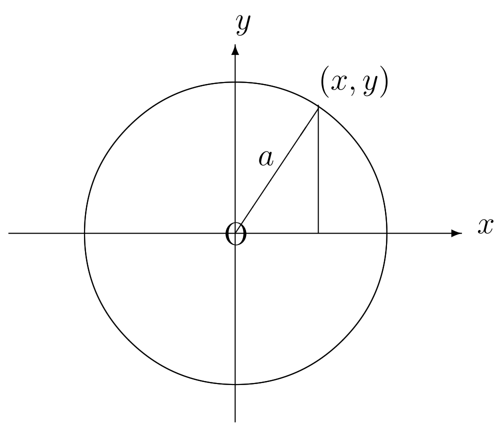
The properties of the four standard conics to be included here will be restricted to those required for simple applications work and, therefore, these notes will not provide an extensive course on elementary co-ordinate geometry.

Useful results from previous work which will be used in these units include the Change of Origin technique (Unit 5.2) and the method of Completing the Square (Unit 1.5). These results should be reviewed, if necessary, by the student.

DEFINITION
A circle is the path traced out by (or **"locus"** of) a point which moves at a fixed distance, called the **"radius"**, from a fixed point, called the **"centre"**.

5.5.2 STANDARD EQUATIONS FOR A CIRCLE

(a) Circle with centre at the origin and having radius a.



Using Pythagoras's Theorem in the diagram, the equation which is satisfied by every point (x, y) on the circle, but no other points in the plane of the axes, is

$$x^2 + y^2 = a^2.$$

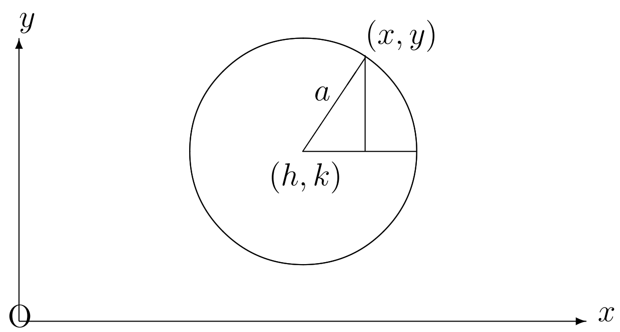
This is therefore the cartesian equation of a circle with centre $(0, 0)$ and radius a.

Note:
The angle θ in the diagram could be used as a parameter for the point (x, y) to give the parametric equations

$$x = a\cos\theta, \quad y = a\sin\theta.$$

Each point on the curve has infinitely many possible parameter values, all differing by a multiple of 2π; but it is usually most convenient to choose the value which lies in the interval $-\pi < \theta \leq \pi$.

(b) Circle with centre (h, k) having radius a.



If we were to consider a temporary change of origin to the point (h, k) with X-axis and Y-axis, the circle would have equation

$$X^2 + Y^2 = a^2,$$

with reference to the new axes. But, from previous work,

$$X = x - h \text{ and } Y = y - k.$$

Hence, with reference to the original axes, the circle has equation

$$(x - h)^2 + (y - k)^2 = a^2;$$

or, in its expanded form,

$$x^2 + y^2 - 2hx - 2ky + c = 0,$$

where

$$c = h^2 + k^2 - a^2.$$

Notes:
(i) The parametric equations of this circle with reference to the temporary new axes through the point (h, k) would be

$$X = a\cos\theta, \quad Y = a\sin\theta.$$

Hence, the parametric equations of the circle with reference to the original axes are

$$x = h + a\cos\theta, \quad y = k + a\sin\theta.$$

(ii) If the equation of a circle is encountered in the form

$$(x-h)^2 + (y-k)^2 = a^2,$$

it is very easy to identify the centre, (h, k) and the radius, a. If the equation is encountered in its expanded form, the best way to identify the centre and radius is to **complete the square in the x and y terms** in order to return to the first form.

EXAMPLES

1. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$x^2 + y^2 + 4x + 6y + 4 = 0.$$

Solution

Completing the square in the x terms,

$$x^2 + 4x \equiv (x+2)^2 - 4.$$

Completing the square in the y terms,

$$y^2 + 6y \equiv (y+3)^2 - 9.$$

The equation of the circle therefore becomes

$$(x+2)^2 + (y+3)^2 = 9.$$

Hence the centre is the point $(-2, -3)$ and the radius is 3.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$5x^2 + 5y^2 - 10x + 15y + 1 = 0.$$

Solution

Here it is best to divide throughout by the coefficient of the x^2 and y^2 terms, even if some of the new coefficients become fractions. We obtain

$$x^2 + y^2 - 2x + 3y + \frac{1}{5} = 0.$$

Completing the square in the x terms,

$$x^2 - 2x \equiv (x-1)^2 - 1.$$

Completing the square in the y terms,

$$y^2 + 3y \equiv \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

The equation of the circle therefore becomes

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{61}{20}.$$

Hence the centre is the point $\left(1, -\frac{3}{2}\right)$ and the radius is $\sqrt{\frac{61}{20}} \cong 1.75$

Note:
Not every equation of the form

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

represents a circle because, for some combinations of h, k and c, the radius would not be a real number. In fact,

$$a = \sqrt{h^2 + k^2 - c},$$

which could easily turn out to be unreal.

5.5.3 EXERCISES

1. Write down the equation of the circle with centre $(4, -3)$ and radius 2.
2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$x^2 + y^2 - 2x + 4y - 11 = 0.$$
3. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$36x^2 + 36y^2 - 36x - 24y - 131 = 0.$$
4. Determine the equation of the circle passing through the point $(4, -3)$ and having centre $(2, 1)$. What is the radius of the circle and what are its parametric equations ?
5. Use the parametric equations of the straight line joining the two points $(2, 4)$ and $-4, 2)$ to find its points of intersection with the circle whose equation is
$$x^2 + y^2 + 4x - 2y = 0.$$

   Hint:
   Substitute the parametric equations of the straight line into the equation of the circle and find two solutions for the parameter.

5.5.4 ANSWERS TO EXERCISES

1. The equation of the circle is either

$$(x-4)^2 + (y+3)^2 = 4,$$

or

$$x^2 + y^2 - 8x + 6y + 21 = 0.$$

2. The centre is $(1, -2)$ and the radius is 4.

3. The centre is $\left(\frac{1}{2}, \frac{1}{3}\right)$ and radius is 2.

4. The equation is

$$(x-2)^2 + (y-1)^2 = 20$$

and the radius is $\sqrt{20}$.
The parametric equations are

$$x = 2 + \sqrt{20}\cos\theta, \quad y = 1 + \sqrt{20}\sin\theta.$$

5. From

$$x = 2 - 6t, \quad \text{and} \quad y = 4 - 2t,$$

we obtain

$$40t^2 - 60t + 20 = 0,$$

giving $t = \frac{1}{2}$ and $t = 1$.
The points of intersection are $(-1, 3)$ and $(-4, 2)$.