"JUST THE MATHS"

UNIT NUMBER

5.3

GEOMETRY 3
(Straight line laws)

by

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UNIT 5.3 - GEOMETRY 3

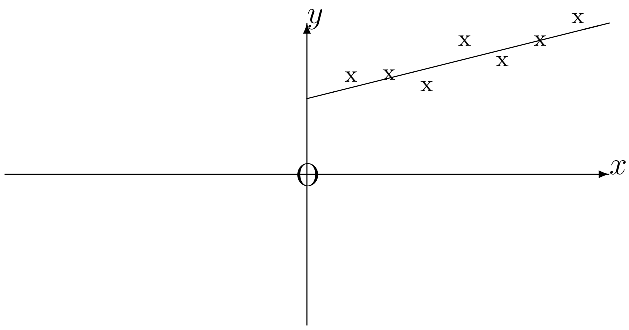
STRAIGHT LINE LAWS

5.3.1 INTRODUCTION

In practical work, the theory of an experiment may show that two variables, x and y, are connected by a straight line equation (or **"straight line law"**) of the form

$$y = mx + c.$$

In order to estimate the values of m and c, we could use the experimental data to plot a graph of y against x and obtain the **"best straight line"** passing through (or near) the plotted points to average out any experimental errors. Points which are obviously out of character with the rest are usually ignored.



It would seem logical, having obtained the best straight line, to measure the gradient, m, and the intercept, c, on the y-axis. However, this not always the wisest way of proceeding and should be avoided in general. The reasons for this are as follows:

(i) Economical use of graph paper may make it impossible to read the intercept, since this part of the graph may be "off the page".

(ii) The use of symbols other than x or y in scientific work may leave doubts as to which is the equivalent of the y-axis and which is the equivalent of the x-axis. Consequently, the gradient may be incorrectly calculated from the graph.

The safest way of finding m and c is to take two sets of readings, (x_1, y_1) and (x_2, y_2), from the best straight line drawn then solve the simultaneous linear equations

$$
\begin{aligned}
y_1 &= mx_1 + c, \\
y_2 &= mx_2 + c.
\end{aligned}
$$

It is a good idea if the two points chosen are as far apart as possible, since this will reduce errors in calculation due to the use of small quantities.

5.3.2 LAWS REDUCIBLE TO LINEAR FORM

Other experimental laws which are not linear can sometimes be reduced to linear form by using the experimental data to plot variables other than x or y, but related to them.

EXAMPLES

1. $y = ax^2 + b$.

   Method

   We let $X = x^2$, so that $y = aX + b$ and hence we may obtain a straight line by plotting y against X.

2. $y = ax^2 + bx$.

   Method

   Here, we need to consider the equation in the equivalent form $\frac{y}{x} = ax + b$ so that, by letting $Y = \frac{y}{x}$, giving $Y = ax + b$, a straight line will be obtained if we plot Y against x.

   Note:

   If one of the sets of readings taken in the experiment happens to be
   $(x, y) = (0, 0)$, we must ignore it in this example.

3. $xy = ax + b$.

   Method

   Two alternatives are available here as follows:

   (a) Letting $xy = Y$, giving $Y = ax + b$, we could plot a graph of Y against x.

   (b) Writing the equation as $y = a + \frac{b}{x}$, we could let $\frac{1}{x} = X$, giving $y = a + bX$, and plot a graph of y against X.

4. $y = ax^b$.

   Method

   This kind of law brings in the properties of logarithms since, if we take logarithms of both sides (base 10 will do here), we obtain

   $$\log_{10} y = \log_{10} a + b \log_{10} x.$$

   Letting $\log_{10} y = Y$ and $\log_{10} x = X$, we have

   $$Y = \log_{10} a + bX,$$

   so that a straight line will be obtained by plotting Y against X.

5. $y = ab^x$.

   Method

   Here again, logarithms may be used to give

   $$\log_{10} y = \log_{10} a + x \log_{10} b$$

   Letting $\log_{10} y = Y$, we have

   $$Y = \log_{10} a + x \log_{10} b,$$

   which will give a straight line if we plot Y against x.

6. $y = ae^{bx}$.

   Method

   In this case, it makes sense to take **natural** logarithms of both sides to give

   $$\log_e y = \log_e a + bx,$$

   which may also be written

   $$\ln y = \ln a + bx$$

   Hence, letting $\ln y = Y$, we can obtain a straight line by plotting a graph of Y against x.

Note:

In all six of the above examples, it is even more important **not** to try to read off the gradient and the intercept from the graph drawn. As before, we should take two sets of readings for x (or X) and y (or Y), substitute them in the straight-line form of the equation and solve two simultaneous linear equations for the constants required.

5.3.3 THE USE OF LOGARITHMIC GRAPH PAPER

In Examples 4,5 and 6 in the previous section, it can be very tedious looking up on a calculator the logarithms of large sets of numbers. We may use, instead, a special kind of graph paper on which there is printed a logarithmic scale (see Unit 1.4) along one or both of the axis directions.

0.1 0.2 0.3 0.4 1 2 3 4 10

Effectively, the logarithmic scale has already looked up the logarithms of the numbers assigned to it provided these numbers are allocated to each **"cycle"** of the scale in successive powers of 10.

Data which includes numbers spread over several different successive powers of ten will need graph paper which has at least that number of cycles in the appropriate axis direction.

For example, the numbers 0.03, 0.09, 0.17, 0.33, 1.82, 4.65, 12, 16, 20, 50 will need **four** cycles on the logarithmic scale.

Accepting these restrictions, which make logarithmic graph paper less economical to use than ordinary graph paper, all we need to do is to plot the **actual** values of the variables whose logarithms we would otherwise have needed to look up. This will give the straight line graph from which we take the usual two sets of readings; these are then substituted into the form of the experimental equation which occurs immediately after taking logarithms of both sides.

It will not matter which base of logarithms is being used since logarithms to two different bases are proportional to each other anyway. The logarithmic graph paper does not, therefore, specify a base.

EXAMPLES

1. $y = ax^b$.

   Method

   (i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + b \log_{10} x$.

   (ii) Plot a graph of y against x, both on logarithmic scales.

   (iii) Estimate the position of the "best straight line".

   (iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + b \log_{10} x_1,\\ \log_{10} y_2 &= \log_{10} a + b \log_{10} x_2.\end{aligned}$$

If it is possible to choose readings which are powers of 10, so much the better, but this is not essential.

2. $y = ab^x$.

Method

(i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + x \log_{10} b$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\log_{10} y_1 &= \log_{10} a + x_1 \log_{10} b,\\ \log_{10} y_2 &= \log_{10} a + x_2 \log_{10} b.\end{aligned}$$

If it is possible to choose zero for the x_1 value, so much the better, but this is not essential.

3. $y = ae^{bx}$.

Method

(i) Taking natural logarithms, $\ln y = \ln a + bx$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned}\ln y_1 &= \ln a + bx_1,\\ \ln y_2 &= \ln a + bx_2.\end{aligned}$$

If it possible to choose zero for the x_1 value, so much the better, but this is not essential.

5.3.5 EXERCISES

In these exercises, use logarithmic graph paper where possible.

1. The following values of x and y can be represented approximately by the law $y = a+bx^2$:

x	0	2	4	6	8	10
y	7.76	11.8	24.4	43.6	71.2	107.0

   Use a straight line graph to find approximately the values of a and b.

2. The following values of x and y are assumed to follow the law $y = ab^x$:

x	0.2	0.4	0.6	0.8	1.4	1.8
y	0.508	0.645	0.819	1.040	2.130	3.420

   Use a straight line graph to find approximately the values of a and b.

3. The following values of x and y are assumed to follow the law $y = ae^{kx}$:

x	0.2	0.5	0.7	1.1	1.3
y	1.223	1.430	1.571	1.921	2.127

   Use a straight line graph to find approximately the values of a and k.

4. The table below gives the pressure, P, and the volume, V, of a certain quantity of steam at maximum density:

P	12.27	17.62	24.92	34.77	47.87	65.06
V	3,390	2,406	1,732	1,264	934.6	699.0

   Assuming that $PV^n = C$, use a straight line graph to find approximately the values of n and C.

5. The coefficient of self induction, L, of a coil, and the number of turns, N, of wire are related by the formula $L = aN^b$, where a and b are constants.
   For the following pairs of observed values, use a straight line graph to calculate approximate values of a and b:

N	25	35	50	75	150	200	250
L	1.09	2.21	5.72	9.60	44.3	76.0	156.0

6. Measurements taken, when a certain gas undergoes compression, give the following values of pressure, p, and temperature, T:

p	10	15	20	25	35	50
T	270	289	303	315	333	353

   Assuming a law of the form $T = ap^n$, use a straight line graph to calculate approximately the values of a and n. Hence estimate the value of T when $p = 32$.

5.3.6 ANSWERS TO EXERCISES

The following answers are approximate; check only that the order of your results are correct. Any slight variations in the position of your straight line could affect the result considerably.

1. $a \simeq 8.0$, $b \simeq 0.99$
2. $a \simeq 0.4$, $b \simeq 3.3$
3. $a \simeq 1.1$, $k \simeq 0.5$
4. $n \simeq 1.06$, $C \simeq 65887$
5. $a \simeq 1.38 \times 10^{-3}$, $b \simeq 2.08$
6. $a \simeq 183.95$, $n \simeq 0.17$