"JUST THE MATHS"

UNIT NUMBER

5.1

GEOMETRY 1
(Co-ordinates, distance & gradient)

by

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UNIT 5.1 - GEOMETRY 1

CO-ORDINATES, DISTANCE AND GRADIENT

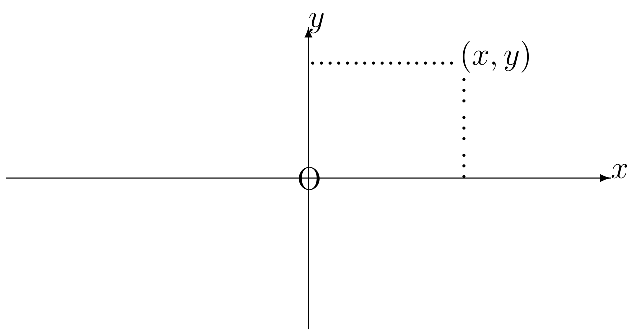
5.1.1 CO-ORDINATES

(a) Cartesian Co-ordinates

The position of a point, P, in a plane may be specified completely if we know its perpendicular distances from two chosen fixed straight lines, where we distinguish between positive distances on one side of each line and negative distances on the other side of each line.

It is not essential that the two chosen fixed lines should be at right-angles to each other, but we usually take them to be so for the sake of convenience.

Consider the following diagram:



The horizontal directed line, Ox, is called the **"x-axis"** and distances to the right of the origin (point O) are taken as positive.

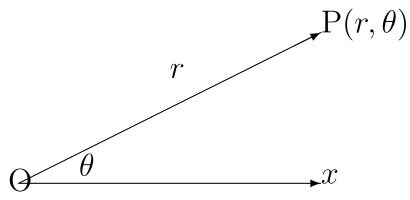
The vertical directed line, Oy, is called the **"y-axis"** and distances above the origin (point O) are taken as positive.

The notation (x, y) denotes a point whose perpendicular distances from Oy and Ox are x and y respectively, these being called the **"cartesian co-ordinates"** of the point.

(b) Polar Co-ordinates

An alternative method of fixing the position of a point P in a plane is to choose first a point, O, called the **"pole"** and directed line , Ox, emanating from the pole in one direction only and called the **"initial line"**.

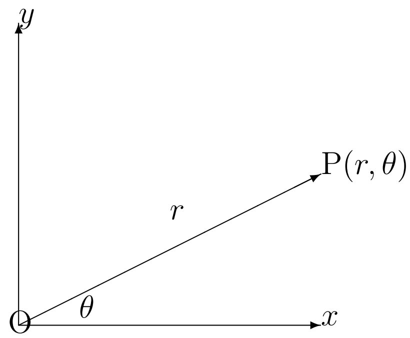
Consider the following diagram:



The position of P is determined by its distance r from the pole and the angle, θ which the line OP makes with the initial line, measuring this angle positively in a counter-clockwise sense or negatively in a clockwise sense from the initial line. The notation (r, θ) denotes the **"polar co-ordinates"** of the point.

5.1.2 THE RELATIONSHIP BETWEEN POLAR AND CARTESIAN CO-ORDINATES

It is convenient to superimpose the diagram for Polar Co-ordinates onto the diagram for Cartesian Co-ordinates as follows:



The trigonometry of the combined diagram shows that

(a) $x = r.\cos\theta$ and $y = r.\sin\theta$;

(b) $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

EXAMPLES

1. Express the equation
$$2x + 3y = 1$$
in polar co-ordinates.
Solution
Substituting for x and y separately, we obtain
$$2r\cos\theta + 3r\sin\theta = 1$$
That is
$$r = \frac{1}{2\cos\theta + 3\sin\theta}$$

2. Express the equation
$$r = \sin\theta$$
in cartesian co-ordinates.

Solution

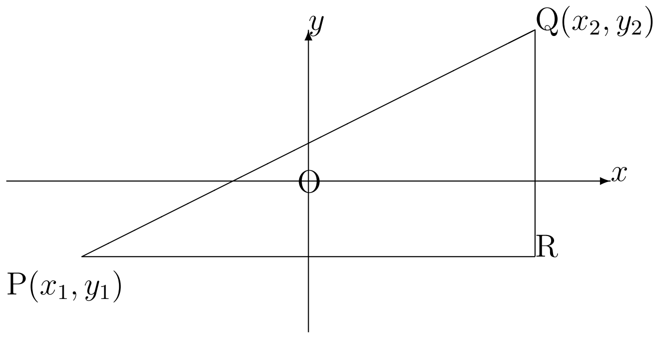
We could try substituting for r and θ separately, but it is easier, in this case, to rewrite the equation as
$$r^2 = r\sin\theta$$
which gives
$$x^2 + y^2 = y$$

5.1.3 THE DISTANCE BETWEEN TWO POINTS

Given two points (x_1, y_1) and (x_2, y_2), the quantity $\mid x_2 - x_1 \mid$ is called the **"horizontal separation"** of the two points and the quantity $\mid y_2 - y_1 \mid$ is called the **"vertical separation"** of the two points, assuming, of course, that the x-axis is horizontal.

The expressions for the horizontal and vertical separations remain valid even when one or more of the co-ordinates is negative. For example, the horizontal separation of the points $(5, 7)$ and $(-3, 2)$ is given by $\mid -3 - 5 \mid = 8$ which agrees with the fact that the two points are on opposite sides of the y-axis.

The actual distance between (x_1, y_1) and (x_2, y_2) is easily calculated from Pythagoras' Theorem, using the horizontal and vertical separations of the points.



In the diagram,
$$\mathrm{PQ}^2 = \mathrm{PR}^2 + \mathrm{RQ}^2.$$
That is,
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$
giving
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note:

We do not need to include the modulus signs of the horizontal and vertical separations because we are squaring them and therefore, any negative signs will disappear. For the same reason, it does not matter which way round the points are labelled.

EXAMPLE
Calculate the distance, d, between the two points $(5, -3)$ and $(-11, -7)$.

Solution

Using the formula, we obtain

$$d = \sqrt{(5 + 11)^2 + (-3 + 7)^2}.$$

That is,

$$d = \sqrt{256 + 16} = \sqrt{272} \cong 16.5$$

5.1.4 GRADIENT

The gradient of the straight-line segment, PQ, joining two points P and Q in a plane is defined to be the tangent of the angle which PQ makes with the positive x-direction.

In practice, when the co-ordinates of the two points are $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$, the gradient, m, is given by either

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$m = \frac{y_1 - y_2}{x_1 - x_2},$$

both giving the same result.

This is not quite the same as the ratio of the horizontal and vertical separations since we distinguish between positive gradient and negative gradient.

EXAMPLE
Determine the gradient of the straight-line segment joining the two points $(8, -13)$ and $(-2, 5)$ and hence calculate the angle which the segment makes with the positive x-direction.

Solution

$$m = \frac{5 + 13}{-2 - 8} = \frac{-13 - 5}{8 + 2} = -1.8$$

Hence, the angle, θ, which the segment makes with the positive x-direction is given by

$$\tan\theta = -1.8$$

Thus,

$$\theta = \tan^{-1}(-1.8) \simeq 119^\circ.$$

5.1.5 EXERCISES

1. A square, side d, has vertices O,A,B,C (labelled counter-clockwise) where O is the pole of a system of polar co-ordinates. Determine the polar co-ordinates of A,B and C when

(a) OA is the initial line;

(b) OB is the initial line.

2. Express the following cartesian equations in polar co-ordinates:

(a)

$$x^2 + y^2 - 2y = 0;$$

(b)

$$y^2 = 4a(a - x).$$

3. Express the following polar equations in cartesian co-ordinates:

(a)

$$r^2 \sin 2\theta = 3;$$

(b)

$$r = 1 + \cos\theta.$$

4. Determine the length of the line segment joining the following pairs of points given in cartesian co-ordinates:

(a) $(0, 0)$ and $(3, 4)$;

(b) $(-2, -3)$ and $(1, 1)$;

(c) $(-4, -6)$ and $(-1, -2)$;

(d) $(2, 4)$ and $(-3, 16)$;

(e) $(-1, 3)$ and $(11, -2)$.

5. Determine the gradient of the straight-line segment joining the two points $(-5, -0.5)$ and $(4.5, -1)$.

5.1.6 ANSWERS TO EXERCISES

1. (a) A$(d, 0)$, B$\left(d\sqrt{2}, \frac{\pi}{4}\right)$, C$\left(d, \frac{\pi}{2}\right)$;

   (b) A$\left(d, -\frac{\pi}{4}\right)$, B$(d\sqrt{2}, 0)$, C$\left(d, \frac{\pi}{4}\right)$.

2. (a) $r = 2\sin\theta$;

   (b) $r^2\sin^2\theta = 4a(a - r\cos\theta)$.

3. (a) $xy = \frac{3}{2}$;

   (b) $x^4 + y^4 + 2x^2y^2 - 2x^3 - 2xy^2 - y^2 = 0$.

4. (a) 5; (b) 5; (c) 5; (d) 13; (e) 13.

5. $m = -\frac{1}{19}$.