"JUST THE MATHS"

UNIT NUMBER

4.2

HYPERBOLIC FUNCTIONS 2
(Inverse hyperbolic functions)

by

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UNIT 4.2 - HYPERBOLIC FUNCTIONS 2
INVERSE HYPERBOLIC FUNCTIONS

4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are $\text{Cosh}^{-1}x$, $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$.

It may be shown that they are given by the following formulae:

(a)

$$\text{Cosh}^{-1}x = \pm\ln(x + \sqrt{x^2 - 1});$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1});$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Notes:

(i) The positive value of $\text{Cosh}^{-1}x$ is called the **'principal value"** and is denoted by $\cosh^{-1}x$ (using a lower-case c).

(ii) $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$ have only **one** value but, for uniformity, we customarily denote them by $\sinh^{-1}x$ and $\tanh^{-1}x$.

4.2.2 THE PROOFS OF THE STANDARD FORMULAE

(a) Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms of both sides gives

$$y = \ln(x \pm \sqrt{x^2 - 1}).$$

However, the two values $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are reciprocals of each other, since their product is the value, 1; and so

$$y = \pm \ln(x + \sqrt{x^2 - 1}).$$

(b) Inverse Hyperbolic Sine

If we let $y = \text{Sinh}^{-1}x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y},$$

or

$$(e^y)^2 - 2xe^y - 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

However, $x - \sqrt{x^2+1}$ has a negative value and cannot, therefore, be equated to a power of e, which is positive. Hence, this part of the expression for e^y must be ignored.

Taking natural logarithms of both sides gives

$$y = \ln(x + \sqrt{x^2+1}).$$

(c) Inverse Hyperbolic Tangent

If we let $y = \text{Tanh}^{-1}x$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

Hence,

$$x\left(e^{2y}+1\right) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1+x}{1-x}.$$

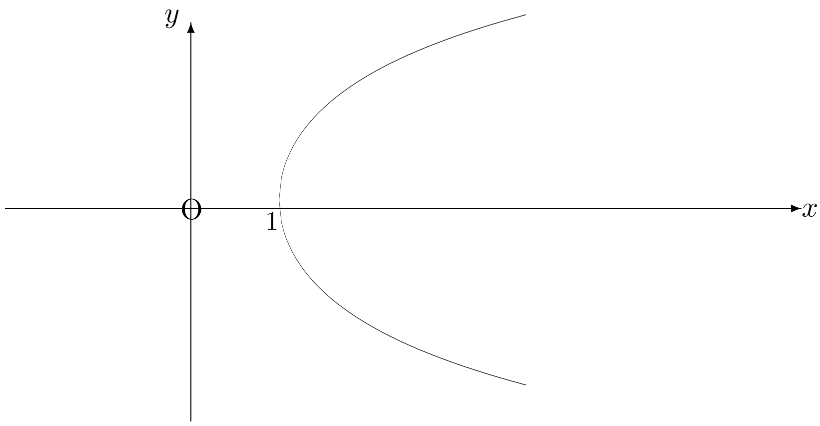
Taking natural logarithms of both sides,

$$y = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

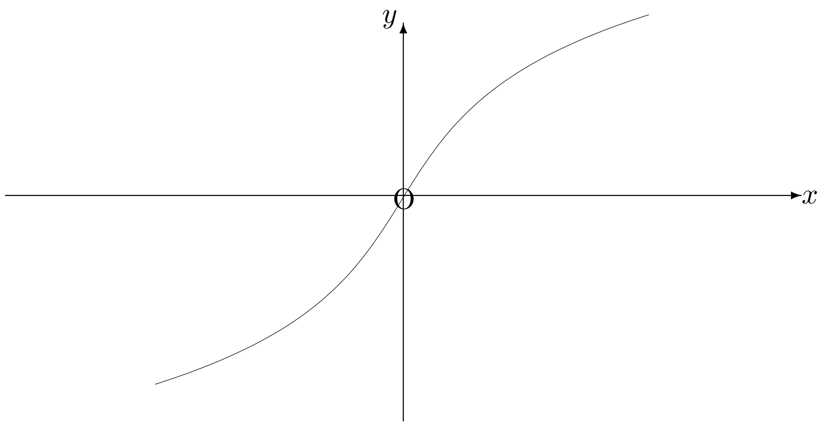
Note:
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

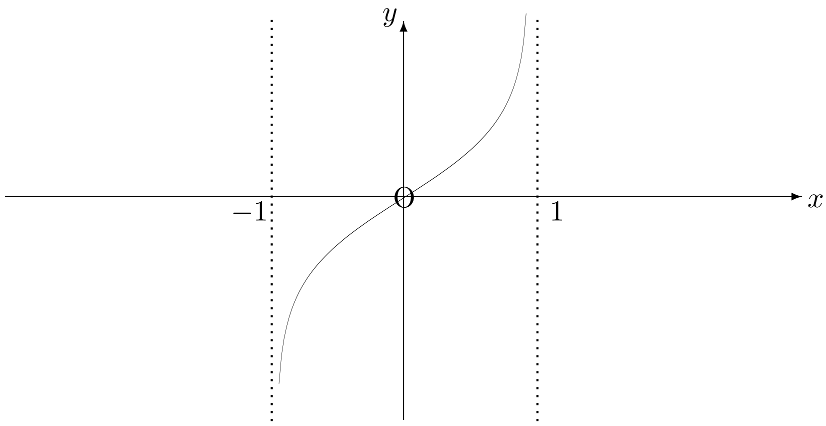
(a) $y = \mathrm{Cosh}^{-1}x$



(b) $y = \mathrm{Sinh}^{-1}x$



(c) $y = \text{Tanh}^{-1}x$



4.2.3 EXERCISES

1. Use the standard formulae to evaluate (a) $\sinh^{-1}7$ and (b) $\cosh^{-1}9$.
2. Express $\cosh 2x$ and $\sinh 2x$ in terms of exponentials and hence solve, for x, the equation

$$2\cosh 2x - \sinh 2x = 2.$$

3. Obtain a formula which equates $\text{cosech}^{-1}x$ to the natural logarithm of an expression in x, distinguishing between the two cases $x > 0$ and $x < 0$.
4. If $t = \tanh(x/2)$, prove that

   (a)

$$\sinh x = \frac{2t}{1-t^2}$$

   and

   (b)

$$\cosh x = \frac{1+t^2}{1-t^2}.$$

   Hence solve, for x, the equation

$$7\sinh x + 20\cosh x = 24.$$

4.2.4 ANSWERS TO EXERCISES

1. (a)

$$\ln(7 + \sqrt{49 + 1}) \simeq 2.644;$$

(b)

$$\ln(9 + \sqrt{81 - 1}) \simeq 2.887$$

2.

$$\left(e^{2x}\right)^2 - 4e^{2x} + 3 = 0,$$

which gives $e^{2x} = 1 \;\text{ or }\; 3$ and hence $x = 0 \;\text{ or }\; \frac{1}{2}\ln 3 \simeq 0.549$.

3. If $x > 0$, then

$$\text{cosech}^{-1}x = \ln\frac{1 + \sqrt{1 + x^2}}{x}.$$

If $x < 0$, then

$$\text{cosech}^{-1}x = \ln\frac{1 - \sqrt{1 + x^2}}{x}.$$

4. Use

$$\sinh x \equiv \frac{2\tanh(x/2)}{\text{sech}^2(x/2)}$$

and

$$\cosh x \equiv \frac{\left(1 + \tanh^2(x/2)\right)}{\text{sech}^2(x/2)}.$$

This gives $t = -\frac{1}{2}$ or $t = \frac{2}{11}$ and hence $x \simeq -1.099 \;\text{ or }\; 0.368$ which agrees with the solution obtained using exponentials.