"JUST THE MATHS"

UNIT NUMBER

4.1

HYPERBOLIC FUNCTIONS 1
(Definitions, graphs and identities)

by

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UNIT 4.1 - HYPERBOLIC FUNCTIONS DEFINITIONS, GRAPHS AND IDENTITIES

4.1.1 INTRODUCTION

In this section, we introduce a new group of mathematical functions, based on the functions

$$e^x \quad \text{and} \quad e^{-x}$$

whose properties resemble, very closely, those of the standard trigonometric functions. But, whereas trigonometric functions can be related to the geometry of a circle (and are sometimes called the **"circular functions"**), it can be shown that the new group of functions are related to the geometry of a hyperbola (see unit 5.7). Because of this, they are called **"hyperbolic functions"**.

4.1.2 DEFINITIONS

(a) Hyperbolic Cosine

The hyperbolic cosine of a number, x, is denoted by $\cosh x$ and is defined by

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"cosh"**.

(b) Hyperbolic Sine

The hyperbolic sine of a number, x, is denoted by $\sinh x$ and is defined by

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"shine"**

(c) Hyperbolic Tangent

The hyperbolic tangent of a number, x, is denoted by $\tanh x$ and is defined by

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

Notes:
(i) The name of the function is pronounced **"than"**.

(ii) In terms of exponentials, it is easily shown that

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(d) Other Hyperbolic Functions

Other, less commonly used, hyperbolic functions are defined as follows:

(i) Hyperbolic secant, pronounced **"shek"**, is defined by

$$\text{sech} x \equiv \frac{1}{\cosh x}.$$

(ii) Hyperbolic cosecant, pronounced **"coshek"** is defined by

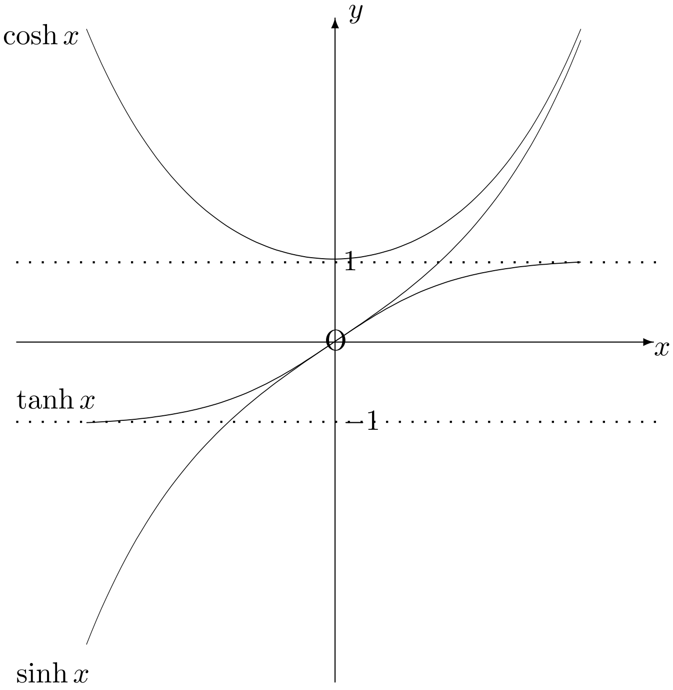
$$\text{cosech} x \equiv \frac{1}{\sinh x}.$$

(iii) Hyperbolic cotangent, pronounced **"coth"** is defined by

$$\text{coth} x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}.$$

4.1.3 GRAPHS OF HYPERBOLIC FUNCTIONS

It is useful to see the graphs of the functions $\cosh x$, $\sinh x$ and $\tanh x$ drawn with reference to the same set of axes. It can be shown that they are as follows:



Note:
We observe that the graph of $\cosh x$ exists only for y greater than or equal to 1; and that graph of $\tanh x$ exists only for y lying between -1 and $+1$. The graph of $\sinh x$, however, covers the whole range of x and y values from $-\infty$ to $+\infty$.

4.1.4 HYPERBOLIC IDENTITIES

It is possible to show that, to every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions though, in some cases, the comparison is more direct than in other cases.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

This follows directly from the definitions of $\cosh x$ and $\sinh x$.

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

Again, this follows from the definitions of $\cosh x$ and $\sinh x$.

3.

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Proof

This follows if we multiply together the results of the previous two illustrations since $e^x.e^{-x} = 1$ and $(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x$.

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity

$$1 - \tanh^2 x \equiv \text{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity

$$\coth^2 x - 1 \equiv \text{cosech}^2 x.$$

4.

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

Proof:

The right hand side may be expressed in the form

$$\frac{e^x - e^{-x}}{2}.\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}.\frac{e^y - e^{-y}}{2},$$

which expands out to

$$\frac{e^{(x+y)} + e^{(x-y)} - e^{(-x+y)} - e^{(-x-y)}}{4} + \frac{e^{(x+y)} - e^{(x-y)} + e^{(-x+y)} - e^{(-x-y)}}{4};$$

and this simplifies to

$$\frac{2e^{(x+y)} - 2e^{(-x-y)}}{4} \equiv \frac{e^{(x+y)} - e^{-(x+y)}}{2} \equiv \sinh(x + y).$$

5.

$$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y.$$

Proof

The proof is similar to the previous illustration.

6.

$$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}.$$

Proof

The proof again is similar to that in Illustration No. 4.

4.1.5 OSBORN'S RULE

Many other results, similar to those previously encountered in the standard list of trigonometric identities can be proved in the same way as for Illustration No. 4 above; that is, we substitute the definitions of the appropriate hyperbolic functions.

However, if we merely wish to **write down** a hyperbolic identity without proving it, we may use the following observation due to Osborn:

Starting with any trigonometric identity, change cos to cosh and sin to sinh. Then, if the trigonometric identity contains (or implies) two sine functions multiplied together, change the sign in front of the relevant term from + to − or vice versa.

ILLUSTRATIONS

1.

$$\cos^2 x + \sin^2 x \equiv 1,$$

which leads to the hyperbolic identity

$$\cosh^2 x - \sinh^2 x \equiv 1$$

since the trigonometric identity contains two sine functions multiplied together.

2.

$$\sin(x-y) \equiv \sin x \cos y - \cos x \sin y,$$

which leads to the hyperbolic identity

$$\sinh(x-y) \equiv \sinh x \cosh y - \cosh x \sinh y$$

in which no changes of sign are required.

3.

$$\sec^2 x \equiv 1 + \tan^2 x,$$

which leads to the hyperbolic identity

$$\text{sech}^2 x \equiv 1 - \tanh^2 x$$

since $\tan^2 x$ in the trigonometric identity implies that two sine functions are multiplied together; that is,

$$\tan^2 x \equiv \frac{\sin^2 x}{\cos^2 x}.$$

4.1.6 EXERCISES

1. If

$$L = 2C \sinh \frac{H}{2C},$$

determine the value of L when $H = 63$ and $C = 50$

2. If

$$v^2 = 1.8L \tanh \frac{6.3d}{L},$$

determine the value of v when $d = 40$ and $L = 315$.

3. Use Osborn's Rule to write down hyperbolic identities for

(a)

$$\sinh 2A;$$

(b)

$$\cosh 2A.$$

4. Use the results of the previous question to simplify the expression

$$\frac{1 + \sinh 2A + \cosh 2A}{1 - \sinh 2A - \cosh 2A}.$$

5. Use Osborn's rule to write down the hyperbolic identity which corresponds to the trigonometric identity

$$2 \sin x \sin y \equiv \cos(x - y) - \cos(x + y)$$

and prove your result.

6. If

$$a = c \cosh x \quad \text{and} \quad b = c \sinh x,$$

show that

$$(a + b)^2 e^{-2x} \equiv a^2 - b^2 \equiv c^2.$$

4.1.7 ANSWERS TO EXERCISES

1. 67.25

2. 19.40

3. (a)

$$\sinh 2A \equiv 2\sinh A\cosh A;$$

(b)

$$\cosh 2A \equiv \cosh^2 A + \sinh^2 A \equiv 2\cosh^2 A - 1 \equiv 1 + 2\sinh^2 A.$$

4.

$$-\coth A.$$

5.

$$-2\sinh x\sinh y \equiv \cosh(x-y) - \cosh(x+y).$$