"JUST THE MATHS"

UNIT NUMBER

3.5

TRIGONOMETRY 5
(Trigonometric identities & wave-forms)

by

A.J.Hobson

UNIT 3.5 - TRIGONOMETRY 5
TRIGONOMETRIC IDENTITIES AND WAVE FORMS

3.5.1 TRIGONOMETRIC IDENTITIES

The standard trigonometric functions can be shown to satisfy a certain group of relationships for any value of the angle θ. They are called **"trigonometric identities"**.

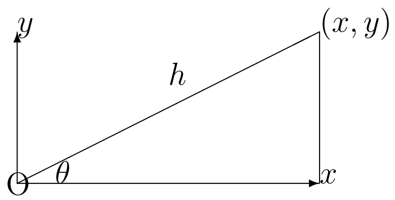
ILLUSTRATION

Prove that

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Proof:

The following diagram was first encountered in Unit 3.1



From the diagram,

$$\cos\theta = \frac{x}{h} \text{ and } \sin\theta = \frac{y}{h}.$$

But, by Pythagoras' Theorem,

$$x^2 + y^2 = h^2.$$

In other words,

$$\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1.$$

That is,

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

It is also worth noting various consequences of this identity:

(a) $\cos^2\theta \equiv 1 - \sin^2\theta$; (rearrangement).

(b) $\sin^2\theta \equiv 1 - \cos^2\theta$; (rearrangement).

(c) $\sec^2\theta \equiv 1 + \tan^2\theta$; (divide by $\cos^2\theta$).

(d) $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$; (divide by $\sin^2\theta$).

Other Trigonometric Identities in common use will not be **proved** here, but they are listed for reference. However, a booklet of Mathematical Formulae should be obtained.

$$\sec\theta \equiv \frac{1}{\cos\theta} \quad \mathrm{cosec}\theta \equiv \frac{1}{\sin\theta} \quad \cot\theta \equiv \frac{1}{\tan\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \quad 1 + \tan^2\theta \equiv \sec^2\theta \quad 1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$$

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin A \equiv 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$$

$$\cos A \equiv \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A \equiv 1 - 2\sin^2 \frac{1}{2} A \equiv 2\cos^2 \frac{1}{2} A - 1$$

$$\tan A \equiv \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A}$$

$$\sin A + \sin B \equiv 2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)$$

$$\sin A - \sin B \equiv 2 \cos \left(\frac{A + B}{2} \right) \sin \left(\frac{A - B}{2} \right)$$

$$\cos A + \cos B \equiv 2 \cos \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)$$

$$\cos A - \cos B \equiv -2 \sin \left(\frac{A + B}{2} \right) \sin \left(\frac{A - B}{2} \right)$$

$$\sin A \cos B \equiv \frac{1}{2} \left[\sin(A + B) + \sin(A - B) \right]$$

$$\cos A \sin B \equiv \frac{1}{2} \left[\sin(A + B) - \sin(A - B) \right]$$

$$\cos A \cos B \equiv \frac{1}{2} \left[\cos(A + B) + \cos(A - B) \right]$$

$$\sin A \sin B \equiv \frac{1}{2} \left[\cos(A - B) - \cos(A + B) \right]$$

$$\sin 3A \equiv 3 \sin A - 4\sin^3 A$$

$$\cos 3A \equiv 4\cos^3 A - 3 \cos A$$

EXAMPLES

1. Show that
$$\sin^2 2x \equiv \frac{1}{2}(1-\cos 4x).$$
Solution
From the standard trigonometric identities, we have
$$\cos 4x \equiv 1 - 2\sin^2 2x$$
on replacing A by $2x$.
Rearranging this new identity, gives the required result.

2. Show that
$$\sin\left(\theta + \frac{\pi}{2}\right) \equiv \cos\theta.$$
Solution
The left hand side can be expanded as
$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2};$$
and the result follows, because $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

3. Simplify the expression
$$\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha}.$$
Solution
Using separate trigonometric identities in the numerator and denominator, the expression becomes
$$\frac{2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\cos\left(\frac{2\alpha-3\alpha}{2}\right)}{-2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\sin\left(\frac{2\alpha-3\alpha}{2}\right)}$$
$$\equiv \frac{2\sin\left(\frac{5\alpha}{2}\right).\cos\left(\frac{-\alpha}{2}\right)}{-2\sin\left(\frac{5\alpha}{2}\right).\sin\left(\frac{-\alpha}{2}\right)}$$
$$\equiv \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$
$$\equiv \cot\left(\frac{\alpha}{2}\right).$$

4. Express $2\sin 3x\cos 7x$ as the difference of two sines.
Solution
$$2\sin 3x\cos 7x \equiv \sin(3x+7x) + \sin(3x-7x).$$
Hence,
$$2\sin 3x\cos 7x \equiv \sin 10x - \sin 4x.$$

3.5.2 AMPLITUDE, WAVE-LENGTH, FREQUENCY AND PHASE ANGLE

In the scientific applications of Mathematics, importance is attached to trigonometric functions of the form

$$A\sin(\omega t+\alpha) \quad \text{and} \quad A\cos(\omega t+\alpha),$$

where A, ω and α are constants and t is usually a time variable.

It is useful to note, from trigonometric identities, that the expanded forms of the above two functions are given by

$$A\sin(\omega t+\alpha) \equiv A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$

and

$$A\cos(\omega t+\alpha) \equiv A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha.$$

(a) The Amplitude

In view of the fact that the sine and the cosine of any angle always lies within the closed interval from -1 to $+1$ inclusive, the constant, A, represents the maximum value (numerically) which can be attained by each of the above trigonometric functions.

A is called the **"amplitude"** of each of the functions.

(b) The Wave Length (Or Period)

If the value, t, increases or decreases by a whole multiple of $\frac{2\pi}{\omega}$, then the value, $(\omega t+\alpha)$, increases or decreases by a whole multiple of 2π; and, hence, the functions remain unchanged in value.

A graph, against t, of either $A\sin(\omega t+\alpha)$ or $A\cos(\omega t+\alpha)$ would be repeated in shape at regular intervals of length $\frac{2\pi}{\omega}$.

The repeated shape of the graph is called the **"wave profile"** and $\frac{2\pi}{\omega}$ is called the **"wave-length"**, or **"period"** of each of the functions.

(c) The Frequency

If t is indeed a time variable, then the wave length (or period) represents the time taken to complete a single wave-profile. Consequently, the number of wave-profiles completed in one unit of time is given by $\frac{\omega}{2\pi}$.

$\frac{\omega}{2\pi}$ is called the **"frequency"** of each of the functions.

Note:
The constant ω itself is called the **"angular frequency"**; it represents the change in the quantity $(\omega t+\alpha)$ for every unit of change in the value of t.

(d) The Phase Angle

The constant, α, affects the starting value, at $t=0$, of the trigonometric functions $A\sin(\omega t+\alpha)$ and $A\cos(\omega t+\alpha)$. Each of these is said to be **"out of phase"**, by an amount, α, with the trigonometric functions $A\sin\omega t$ and $A\cos\omega t$ respectively.

α is called the **"phase angle"** of each of the two original trigonometric functions; but it can take infinitely many values differing only by a whole multiple of 360° (if working in degrees) or 2π (if working in radians).

EXAMPLES

1. Express $\sin t + \sqrt{3}\cos t$ in the form $A\sin(t + \alpha)$, with α in degrees, and hence solve the equation,

$$\sin t + \sqrt{3}\cos t = 1$$

for t in the range $0^\circ \leq t \leq 360^\circ$.

Solution

We require that

$$\sin t + \sqrt{3}\cos t \equiv A\sin t\cos\alpha + A\cos t\sin\alpha$$

Hence,

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = \sqrt{3},$$

which gives $A^2 = 4$ (using $\cos^2\alpha + \sin^2\alpha \equiv 1$) and also $\tan\alpha = \sqrt{3}$.

Thus,

$$A = 2 \quad \text{and} \quad \alpha = 60^\circ \text{ (principal value).}$$

To solve the given equation, we may now use

$$2\sin(t + 60^\circ) = 1,$$

so that

$$t + 60^\circ = \text{Sin}^{-1}\frac{1}{2} = 30^\circ + k360^\circ \quad \text{or} \quad 150^\circ + k360^\circ,$$

where k may be any integer.

For the range $0^\circ \leq t \leq 360^\circ$, we conclude that

$$t = 330^\circ \quad \text{or} \quad 90^\circ.$$

2. Determine the amplitude and phase-angle which will express the trigonometric function $a\sin\omega t + b\cos\omega t$ in the form $A\sin(\omega t + \alpha)$.

Apply the result to the expression $3\sin 5t - 4\cos 5t$ stating α in degrees, correct to one decimal place, and lying in the interval from -180° to 180°.

Solution

We require that

$$A\sin(\omega t + \alpha) \equiv a\sin\omega t + b\cos\omega t;$$

and, hence, from trigonometric identities,

$$A\sin\alpha = b \quad \text{and} \quad A\cos\alpha = a.$$

Squaring each of these and adding the results together gives

$$A^2 = a^2 + b^2 \quad \text{that is} \quad A = \sqrt{a^2 + b^2}.$$

Also,

$$\frac{A\sin\alpha}{A\cos\alpha} = \frac{b}{a},$$

which gives

$$\alpha = \tan^{-1}\frac{b}{a};$$

but the particular angle chosen must ensure that $\sin\alpha = \frac{b}{A}$ and $\cos\alpha = \frac{a}{A}$ have the correct sign.

Applying the results to the expression $3\sin 5t - 4\cos 5t$, we have

$$A = \sqrt{3^2 + 4^2}$$

and

$$\alpha = \tan^{-1}\left(-\frac{4}{3}\right).$$

But $\sin\alpha \left(= -\frac{4}{5}\right)$ is negative and $\cos\alpha \left(= \frac{3}{5}\right)$ is positive so that α may be taken as an angle between zero and $-90°$; that is $\alpha = -53.1°$.

We conclude that

$$3\sin 5t - 4\cos 5t \equiv 5\sin(5t - 53.1°).$$

3. Solve the equation

$$4\sin 2t + 3\cos 2t = 1$$

for t in the interval from $-180°$ to $180°$.

Solution

Expressing the left hand side of the equation in the form $A\sin(2t + \alpha)$, we require

$$A = \sqrt{4^2 + 3^2} = 5 \quad \text{and} \quad \alpha = \tan^{-1}\frac{3}{4}.$$

Also $\sin\alpha \left(= \frac{3}{5}\right)$ is positive and $\cos\alpha \left(= \frac{4}{5}\right)$ is positive so that α may be taken as an angle in the interval from zero to $90°$.

Hence, $\alpha = 36.87°$ and the equation to be solved becomes

$$5\sin(2t + 36.87°) = 1.$$

Its solutions are obtained by making t the **"subject"** of the equation to give

$$t = \frac{1}{2}\left[\text{Sin}^{-1}\frac{1}{5} - 36.87°\right].$$

The possible values of $\text{Sin}^{-1}\frac{1}{5}$ are $11.53° + k360°$ and $168.46° + k360°$, where k may be any integer. But, to give values of t which are numerically less than $180°$, we may use only $k = 0$ and $k = 1$ in the first of these and $k = 0$ and $k = -1$ in the second.

The results obtained are

$$t = -12.67°,\ 65.80°,\ 167.33° \quad \text{and} \quad -114.21°$$

3.5.3 EXERCISES

1. Simplify the following expressions:

   (a)
$$(1 + \cos x)(1 - \cos x);$$

   (b)
$$(1 + \sin x)^2 - 2\sin x(1 + \sin x).$$

2. Show that
$$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \sin\theta$$

3. Express $2\sin 4x \sin 5x$ as the difference of two cosines.

4. Use the table of trigonometric identities to show that

   (a)
$$\frac{\sin 5x + \sin x}{\cos 5x + \cos x} \equiv \tan 3x;$$

   (b)
$$\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x;$$

   (c)
$$\tan x. \tan 2x + 2 \equiv \tan 2x. \cot x;$$

   (d)
$$\cot(x + y) \equiv \frac{\cot x. \cot y - 1}{\cot y + \cot x}.$$

5. Solve the following equations by writing the trigonometric expression on the left-hand-side in the form suggested, being careful to see whether the phase angle is required in degrees or radians and ensuring that your final answers are in the range given:

   (a) $\cos t + 7\sin t = 5, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t - \alpha)$, with α in degrees.
   (b) $\sqrt{2}\cos t - \sin t = 1, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t + \alpha)$, with α in degrees.
   (c) $2\sin t - \cos t = 1, \quad 0 \leq t \leq 2\pi$, (transposed to the form $A\sin(t - \alpha)$, with α in radians.
   (d) $3\sin t - 4\cos t = 0.6, \quad 0° \leq t \leq 360°$, (transposed to the form $A\sin(t - \alpha)$, with α in degrees.

6. Determine the amplitude and phase-angle which will express the trigonometric function $a\cos\omega t + b\sin\omega t$ in the form $A\cos(\omega t + \alpha)$.

   Apply the result to the expression $4\cos 5t - 4\sqrt{3}\sin 5t$ stating α in degrees and lying in the interval from $-180°$ to $180°$.

7. Solve the equation

$$2\cos 3t + 5\sin 3t = 4$$

for t in the interval from zero to 360°, expressing t in decimals correct to two decimal places.

3.5.4 ANSWERS TO EXERCISES

1. (a) $\sin^2 x$; (b) $\cos^2 x$.
2. Use the cos(A - B) formula to expand left hand side.
3. $\cos x - \cos 9x$.
4. (a) Use the formulae for sinA + sinB and cosA + cosB;
   (b) Use the formulae for $\cos 2x$ to make the 1's cancel;
   (c) Both sides are identically equal to $\frac{2}{1-\tan^2 x}$;
   (d) Invert the formula for $\tan(x+y)$.
5. (a) 36.9°, 126.9°;
   (b) 19.5°, 270°;
   (c) 0, 3.14;
   (d) 226.24°
6.

$$A = \sqrt{a^2+b^2}, \quad \text{and} \quad \alpha = \tan^{-1}\left(-\frac{b}{a}\right);$$

$$4\cos 5t - 4\sqrt{3}\sin 5t \equiv 8\cos(5t + 60°).$$

7.

$$\sqrt{29}\cos(3t - 68.20°) = 4 \quad \text{or} \quad \sqrt{29}\sin(3t + 21.80°) = 4$$

give

$$t = 8.72°,\ 36.74°,\ 156.74° \quad \text{and} \quad 276.74°$$