"JUST THE MATHS"

UNIT NUMBER

3.4

TRIGONOMETRY 4
(Solution of triangles)

by

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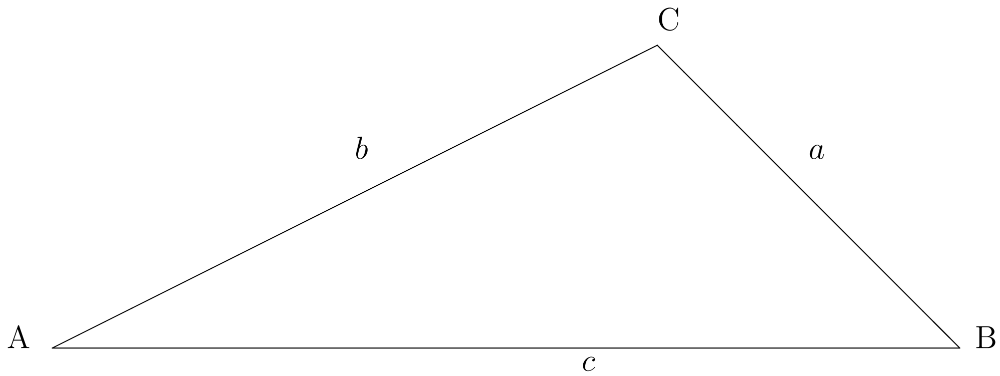
UNIT 3.4 - TRIGONOMETRY 4
SOLUTION OF TRIANGLES

3.4.1 INTRODUCTION

The **"solution of a triangle"** is defined to mean the complete set of data relating to the lengths of its three sides and the values of its three interior angles. It can be shown that these angles always add up to 180°.

If a sufficient amount of information is provided about **some** of this data, then it is usually possible to determine the remaining data.

We shall use a standardised type of diagram for an arbitrary triangle whose **"vertices"** (i.e. corners) are A,B and C and whose sides have lengths a, b and c. It is as follows:



The angles at A,B and C will be denoted by $\widehat{A}, \widehat{B}$ and $\widehat{C}$.

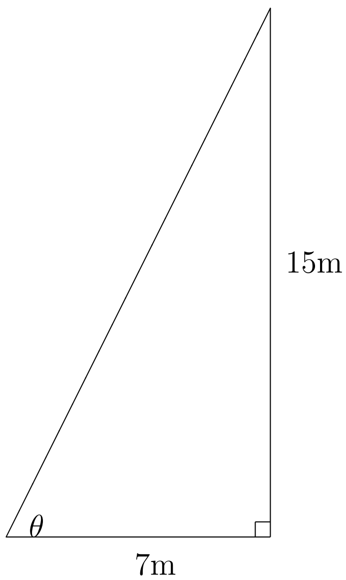
3.4.2 RIGHT-ANGLED TRIANGLES

Right-angled triangles are easier to solve than the more general kinds of triangle because all we need to use are the relationships between the lengths of the sides and the trigonometric ratios sine, cosine and tangent. An example will serve to illustrate the technique:

EXAMPLE

From the top of a vertical pylon, 15 meters high, a guide cable is to be secured into the (horizontal) ground at a distance of 7 meters from the base of the pylon.
What will be the length of the cable and what will be its inclination (in degrees) to the horizontal ?

Solution



From Pythagoras' Theorem, the length of the cable will be

$$\sqrt{7^2 + 15^2} \simeq 16.55\text{m}.$$

The angle of inclination to the horizontal will be θ, where

$$\tan\theta = \frac{15}{7}.$$

$$\text{Hence,} \quad \theta \simeq 65^\circ.$$

3.4.3 THE SINE AND COSINE RULES

Two powerful tools for the solution of triangles in general may be stated in relation to the earlier diagram as follows:

(a) The Sine Rule

$$\frac{a}{\sin \widehat{A}} = \frac{b}{\sin \widehat{B}} = \frac{c}{\sin \widehat{C}}.$$

(b) The Cosine Rule

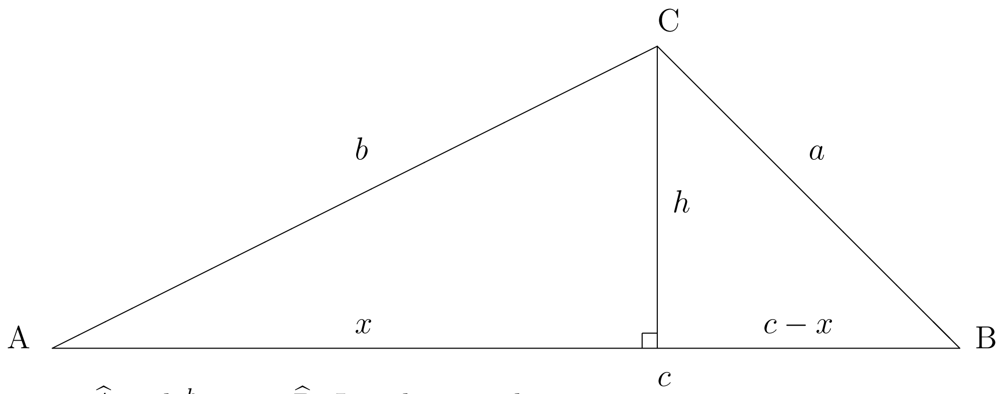
$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A};$$
$$b^2 = c^2 + a^2 - 2ca \cos \widehat{B};$$
$$c^2 = a^2 + b^2 - 2ab \cos \widehat{C}.$$

Clearly, the last two of these are variations of the first.

We also observe that, whenever the angle on the right-hand-side is a right-angle, the Cosine Rule reduces to Pythagoras' Theorem.

The Proof of the Sine Rule

In the diagram encountered earlier, suppose we draw the perpendicular (of length h) from the vertex C onto the side AB.



Then $\frac{h}{b} = \sin\widehat{A}$ and $\frac{h}{a} = \sin\widehat{B}$. In other words,

$$b\sin\widehat{A} = a\sin\widehat{B}$$

or

$$\frac{b}{\sin\widehat{B}} = \frac{a}{\sin\widehat{A}}.$$

Clearly, the remainder of the Sine Rule can be obtained by considering the perpendicular drawn from a different vertex.

The Proof of the Cosine Rule

Using the same diagram as for the Sine Rule, we can assume that the side AB has lengths x and $c-x$ either side of the foot of the perpendicular drawn from C. Hence

$$h^2 = b^2 - x^2$$

and, at the same time,

$$h^2 = a^2 - (c-x)^2.$$

Expanding and equating the two expressions for h^2, we obtain

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$

that is

$$a^2 = b^2 + c^2 - 2xc.$$

But $x = b\cos\widehat{A}$, and so

$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A}.$$

EXAMPLES

1. Solve the triangle ABC in the case when $\widehat{A} = 20°, \widehat{B} = 30°$ and $c = 10$cm.

   Solution

   Firstly, the angle $\widehat{C} = 130°$ since the interior angles must add up to $180°$.

   Thus, by the Sine Rule, we have

$$\frac{a}{\sin 20°} = \frac{b}{\sin 30°} = \frac{10}{\sin 130°}.$$

   That is,

$$\frac{a}{0.342} = \frac{b}{0.5} = \frac{10}{0.766}$$

   These give the results

$$a = \frac{10 \times 0.342}{0.766} \cong 4.47\text{cm}$$
$$b = \frac{10 \times 0.5}{0.766} \cong 6.53\text{cm}$$

2. Solve the triangle ABC in the case when $b = 9\text{cm}, c = 5\text{cm}$ and $\widehat{A} = 70°$.

   Solution

   In this case, the information prevents us from using the Sine Rule immediately, but the Cosine Rule **can** be applied as follows:

$$a^2 = 25 + 81 - 90 \cos 70°$$

   giving

$$a^2 = 106 - 30.782 = 75.218$$

   Hence

$$a \simeq 8.673\text{cm} \simeq 8.67\text{cm}$$

   Now we can use the Sine Rule to complete the solution

$$\frac{8.673}{\sin 70°} = \frac{9}{\sin \widehat{B}} = \frac{5}{\sin \widehat{C}}.$$

   Thus,

$$\sin \widehat{B} = \frac{9 \times \sin 70°}{8.673} = \frac{9 \times 0.940}{8.673} \simeq 0.975$$

   This suggests that $\widehat{B} \simeq 77.19°$ in which case $\widehat{C} \simeq 180° - 70° - 77.19° \simeq 32.81°$ but, for the moment, we must also allow the possibility that $\widehat{B} \simeq 102.81°$ which would give $\widehat{C} \simeq 7.19°$

   However, we can show that the alternative solution is unacceptable because it is not consistent with the whole of the Sine Rule statement for this example. Thus the only solution is the one for which

$$a \simeq 8.67\text{cm}, \quad \widehat{B} \simeq 77.19°, \quad \widehat{C} \simeq 32.81°$$

Note: It is possible to encounter examples for which more than one solution **does** exist.

3.4.4 EXERCISES

Solve the triangle ABC in the following cases:

1. $c = 25\text{cm}$, $\widehat{A} = 35°$, $\widehat{B} = 68°$.
2. $c = 23\text{cm}$, $a = 30\text{cm}$, $\widehat{C} = 40°$.
3. $b = 4\text{cm}$, $c = 5\text{cm}$, $\widehat{A} = 60°$.
4. $a = 21\text{cm}$, $b = 23\text{cm}$, $c = 16\text{cm}$.

3.4.5 ANSWERS TO EXERCISES

1. $a \simeq 14.72\text{cm}$, $b \simeq 23.79\text{cm}$, $\widehat{C} \simeq 77°$.
2. $\widehat{A} \simeq 56.97°$, $\widehat{B} \simeq 83.03°$, $b = 35.52\text{cm}$;
   OR
   $\widehat{A} \simeq 123.03°$, $\widehat{B} \simeq 16.97°$, $b \simeq 10.44\text{cm}$.
3. $a \simeq 4.58\text{cm}$, $\widehat{B} \simeq 49.11°$, $\widehat{C} \simeq 70.89°$.
4. $\widehat{A} \simeq 62.13°$, $\widehat{B} \simeq 75.52°$, $\widehat{C} \simeq 42.35°$.