"JUST THE MATHS"

UNIT NUMBER

3.3

TRIGONOMETRY 3
(Approximations & inverse functions)

by

A.J.Hobson

UNIT 3.3 - TRIGONOMETRY
APPROXIMATIONS AND INVERSE FUNCTIONS

3.3.1 APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS

Three standard approximations for the functions $\sin\theta, \cos\theta$ and $\tan\theta$ respectively can be obtained from a set of results taken from the applications of Calculus. These are stated without proof as follows:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}.....$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}......$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + ...$$

These results apply **only if θ is in radians** but, if θ is small enough for θ^2 and higher powers of θ to be neglected, we conclude that

$$\sin\theta \simeq \theta,$$

$$\cos\theta \simeq 1,$$

$$\tan\theta \simeq \theta.$$

Better approximations are obtainable if more terms of the infinite series are used.

EXAMPLE
Approximate the function

$$5 + 2\cos\theta - 7\sin\theta$$

to a quartic polynomial in θ.

Solution

Using terms of the appropriate series up to and including the fourth power of θ, we deduce that

$$5 + 2\cos\theta - 7\sin\theta \simeq 5 + 2 - \theta^2 + \frac{\theta^4}{12} - 7\theta + 7\frac{\theta^3}{6}$$

$$= \frac{1}{12}\left[\theta^4 + 14\theta^3 - 12\theta^2 - 84\theta + 84\right].$$

3.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

It is frequently necessary to determine possible angles for which the value of their sine, cosine or tangent is already specified. This is carried out using inverse trigonometric functions defined as follows:

(a) The symbol

$$\text{Sin}^{-1}x$$

denotes any angle whose sine value is the number x. It is necessary that $-1 \leq x \leq 1$ since the sine of an angle is always in this range.

(b) The symbol

$$\text{Cos}^{-1}x$$

denotes any angle whose cosine value is the number x. Again, $-1 \leq x \leq 1$.

(c) The symbol

$$\text{Tan}^{-1}x$$

denotes any angle whose tangent value is x. This time, x may be any value because the tangent function covers the range from $-\infty$ to ∞.

We note that because of the **A** ll, **S** ine, **T** angent, **C** osine diagram, (see Unit 3.1), there will be two **basic** values of an inverse function from two different quadrants. But either of these two values may be increased or decreased by a whole multiple of 360° (2π) to yield other acceptable answers and hence an infinite number of possible answers.

EXAMPLES

1. Evaluate $\text{Sin}^{-1}(\frac{1}{2})$.

   Solution

   $\text{Sin}^{-1}(\frac{1}{2}) = 30° \pm n360°$ or $150° \pm n360°$.

2. Evaluate $\text{Tan}^{-1}(\sqrt{3})$.

   Solution

   $\text{Tan}^{-1}(\sqrt{3}) = 60° \pm n360°$ or $240° \pm n360°$.

   This result is in fact better written in the combined form
   $\text{Tan}^{-1}(\sqrt{3}) = 60° \pm n180°$
   THat is, angles in opposite quadrants have the same tangent.

   Another Type of Question

3. Obtain all of the solutions to the equation

$$\cos 3x = -0.432$$

   which lie in the interval $-180° \leq x \leq 180°$.

   Solution

   This type of question is of a slightly different nature since we are asked for a specified **selection** of values rather than the general solution of the equation.

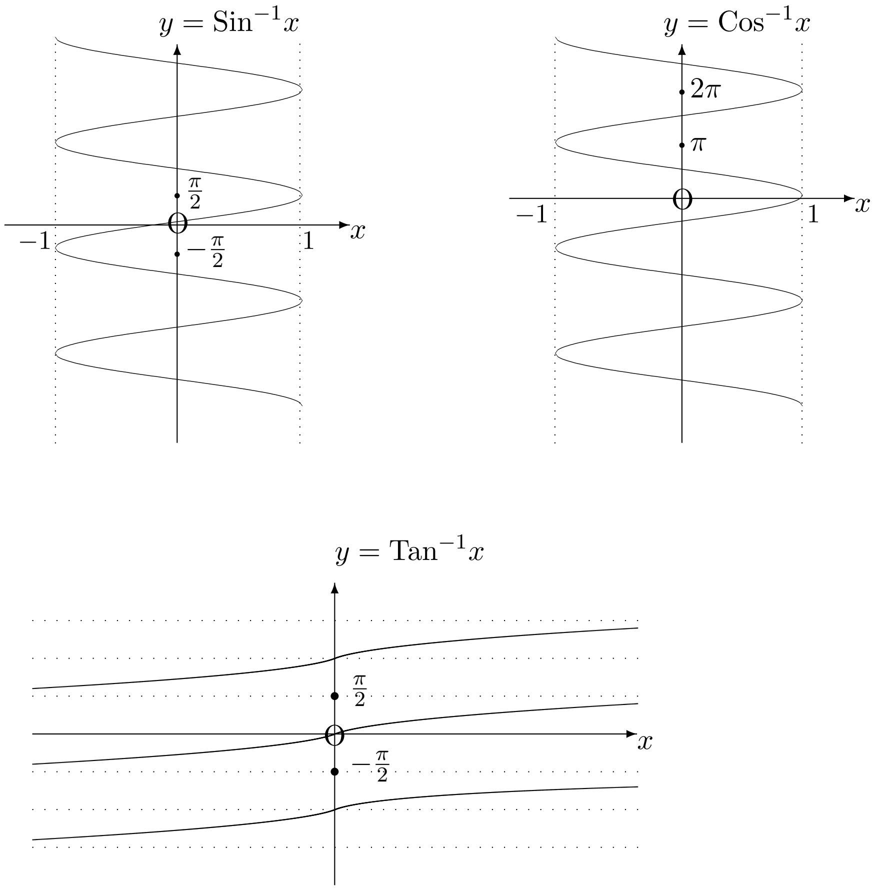
We require that $3x$ be any one of the angles (within an interval $-540° \leq 3x \leq 540°$) whose cosine is equal to -0.432. Using a calculator, the simplest angle which satisfies this condition is $115.59°$; but the complete set is

$$\pm 115.59° \quad \pm 244.41° \quad \pm 475.59°$$

Thus, on dividing by 3, the possibilities for x are

$$\pm 38.5° \quad \pm 81.5° \quad \pm 158.5°$$

Note: The graphs of inverse trigonometric functions are discussed fully in Unit 10.6, but we include them here for the sake of completeness



Of all the possible values obtained for an inverse trigonometric function, one particular one is called the **"Principal Value"**. It is the unique value which lies in a specified range described below, the explanation of which is best dealt with in connection with differential calculus.

To indicate such a principal value, we use the lower-case initial letter of each inverse function.

(a) $\theta = \sin^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b) $\theta = \cos^{-1}x$ lies in the range $0 \leq \theta \leq \pi$.

(c) $\theta = \tan^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

EXAMPLES

1. Evaluate $\sin^{-1}(\frac{1}{2})$.

   Solution

   $\sin^{-1}(\frac{1}{2}) = 30^\circ$ or $\frac{\pi}{6}$.

2. Evaluate $\tan^{-1}(-\sqrt{3})$.

   Solution

   $\tan^{-1}(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$.

3. Write down a formula for u in terms of v in the case when

   $$v = 5\cos(1 - 7u).$$

   Solution

   Dividing by 5 gives

   $$\frac{v}{5} = \cos(1 - 7u).$$

   Taking the inverse cosine gives

   $$\text{Cos}^{-1}\left(\frac{v}{5}\right) = 1 - 7u.$$

   Subtracting 1 from both sides gives

   $$\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1 = -7u.$$

   Dividing both sides by -7 gives

   $$u = -\frac{1}{7}\left[\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1\right].$$

3.3.3 EXERCISES

1. If powers of θ higher then three can be neglected, find an approximation for the function

   $$6\sin\theta + 2\cos\theta + 10\tan\theta$$

   in the form of a polynomial in θ.

2. If powers of θ higher than five can be neglected, find an approximation for the function

   $$2\sin\theta - \theta\cos\theta$$

   in the form of a polynomial in θ.

3. If powers of θ higher than two can be neglected, show that the function

$$\frac{\theta \sin \theta}{1 - \cos \theta}$$

is approximately equal to 2.

4. Write down the principal values of the following:

   (a) $\mathrm{Sin}^{-1} 1$;
   (b) $\mathrm{Sin}^{-1}\left(-\frac{1}{2}\right)$;
   (c) $\mathrm{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$;
   (d) $\mathrm{Tan}^{-1} 5$;
   (e) $\mathrm{Tan}^{-1}(-\sqrt{3})$;
   (f) $\mathrm{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

5. Solve the following equations for x in the interval $0 \leq x \leq 360°$:

   (a)
   $$\tan x = 2.46$$
   (b)
   $$\cos x = 0.241$$
   (c)
   $$\sin x = -0.786$$
   (d)
   $$\tan x = -1.42$$
   (e)
   $$\cos x = -0.3478$$
   (f)
   $$\sin x = 0.987$$

   Give your answers correct to one decimal place.

6. Solve the following equations for the range given, stating your final answers in degrees correct to one decimal place:

   (a) $\sin 2x = -0.346$ for $0 \leq x \leq 360°$;
   (b) $\tan 3x = 1.86$ for $0 \leq x \leq 180°$;
   (c) $\cos 2x = -0.57$ for $-180° \leq x \leq 180°$;
   (d) $\cos 5x = 0.21$ for $0 \leq x \leq 45°$;
   (e) $\sin 4x = 0.78$ for $0 \leq x \leq 180°$.

7. Write down a formula for u in terms of v for the following:

   (a) $v = \sin u$;
   (b) $v = \cos 2u$;
   (c) $v = \tan(u + 1)$.

8. If x is positive, show diagramatically that

(a)

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};$$

(b)

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}.$$

3.3.4 ANSWERS TO EXERCISES

1. $\frac{7\theta^3}{3} - \theta^2 + 16\theta + 2.$
2. $\theta + \frac{\theta^3}{6} - \frac{\theta^5}{40}.$
3. Substitute approximations for $\sin\theta$ and $\cos\theta$.
4. (a) $\frac{\pi}{2}$; (b) $-\frac{\pi}{6}$; (c) $\frac{5\pi}{6}$; (d) 1.373; (e) $-\frac{\pi}{3}$; (f) $\frac{3\pi}{4}$.
5. (a) 67.9° or 247.9°;
   (b) 76.1° or 283.9°;
   (c) 231.8° or 308.2°;
   (d) 125.2° or 305.32°;
   (e) 110.4° or 249.6°;
   (f) 80.8° or 99.2°
6. (a) 100.1°, 169.9°, 280.1°, 349.9°
   (b) 20.6°, 80.6°, 140.6°
   (c) $\pm 62.4^\circ$, $\pm 117.6^\circ$
   (d) 15.6°
   (e) 32.2°, 102.8°, 122.2°
7. (a) $u = \text{Sin}^{-1}v$; (b) $u = \frac{1}{2}\text{Cos}^{-1}v$; (c) $u = \text{Tan}^{-1}v - 1$.
8. A suitable diagram is