“JUST THE MATHS”

UNIT NUMBER

3.1

TRIGONOMETRY 1
(Angles & trigonometric functions)

by

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UNIT 3.1 - TRIGONOMETRY 1
ANGLES AND TRIGONOMETRIC FUNCTIONS

3.1.1 INTRODUCTION

The following results will be assumed without proof:

(i) The Circumference, C, and Diameter, D, of a circle are directly proportional to each other through the formula

$$C = \pi D$$

or, if the radius is r,

$$C = 2\pi r.$$

(ii) The area, A, of a circle is related to the radius, r, by means of the formula

$$A = \pi r^2.$$

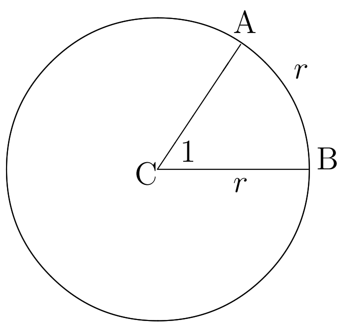
3.1.2 ANGULAR MEASURE

(a) Astronomical Units

The **"degree"** is a $\frac{1}{360}$th part of one complete revolution. It is based on the study of planetary motion where 360 is approximately the number of days in a year.

(b) Radian Measure

A **"radian"** is the angle subtended at the centre of a circle by an arc which is equal in length to the radius.



RESULTS

(i) Using the definition of a radian, together with the second formula for circumference on the previous page, we conclude that there are 2π radians in one complete revolution. That is, 2π radians is equivalent to $360°$ or, in other words π radians is equivalent to $180°$.

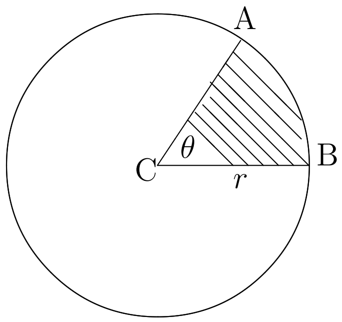
(ii) In the diagram overleaf, the arclength from A to B will be given by

$$\frac{\theta}{2\pi} \times 2\pi r = r\theta,$$

assuming that θ is measured in radians.

(iii) In the diagram below, the area of the sector ABC is given by

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$



(c) Standard Angles

The scaling factor for converting degrees to radians is

$$\frac{\pi}{180}$$

and the scaling factor for converting from radians to degrees is

$$\frac{180}{\pi}.$$

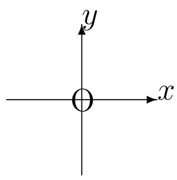
These scaling factors enable us to deal with any angle, but it is useful to list the expression, in radians, of some of the more well-known angles.

ILLUSTRATIONS

1. $15°$ is equivalent to $\frac{\pi}{180} \times 15 = \frac{\pi}{12}$.
2. $30°$ is equivalent to $\frac{\pi}{180} \times 30 = \frac{\pi}{6}$.
3. $45°$ is equivalent to $\frac{\pi}{180} \times 45 = \frac{\pi}{4}$.
4. $60°$ is equivalent to $\frac{\pi}{180} \times 60 = \frac{\pi}{3}$.
5. $75°$ is equivalent to $\frac{\pi}{180} \times 75 = \frac{5\pi}{12}$.
6. $90°$ is equivalent to $\frac{\pi}{180} \times 90 = \frac{\pi}{2}$.

(d) Positive and Negative Angles

For the measurement of angles in general, we consider the plane of the page to be divided into four quadrants by means of a cartesian reference system with axes Ox and Oy. The **"first quadrant"** is that for which x and y are both positive, and the other three quadrants are numbered from the first in an anticlockwise sense.



From the positive x-direction, we measure angles positively in the anticlockwise sense and negatively in the clockwise sense. Special names are given to the type of angles obtained as follows:

1. Angles in the range between 0° and 90° are called **"positive acute"** angles.

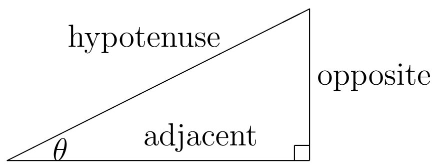
2. Angles in the range between 90° and 180° are called **"positive obtuse"** angles.

3. Angles in the range between 180° and 360° are called **"positive reflex"** angles.

4. Angles measured in the clockwise sense have similar names but preceded by the word **"negative"**.

3.1.3 TRIGONOMETRIC FUNCTIONS

We first consider a right-angled triangle in one corner of which is an angle θ other than the right-angle itself. The sides of the triangle are labelled in relation to this angle , θ, as **"opposite"**, **"adjacent"** and **"hypotenuse"** (see diagram below).



For future reference, we shall assume, without proof, the result known as **"Pythagoras' Theorem"**. This states that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

DEFINITIONS

(a) The **"sine"** of the angle θ, denoted by $\sin\theta$, is defined by

$$\sin\theta \equiv \frac{\text{opposite}}{\text{hypotenuse}};$$

(b) The **"cosine"** of the angle θ, denoted by $\cos\theta$, is defined by

$$\cos\theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}};$$

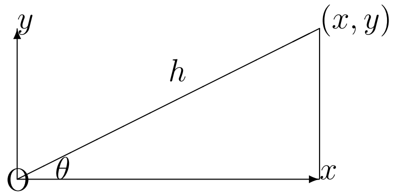
(c) The **"tangent"** of the angle θ, denoted by $\tan\theta$, is defined by

$$\tan\theta \equiv \frac{\text{opposite}}{\text{adjacent}}.$$

Notes:
(i) The traditional aid to remembering the above definitions is the abbreviation

S.O.H.C.A.H.T.O.A.

(ii) The definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be extended to angles of any size by regarding the end-points of the hypotenuse, with length h, to be, respectively, the origin and the point (x, y) in a cartesian system of reference.



For any values of x and y, positive, negative or zero, the three basic trigonometric functions are defined in general by the formulae

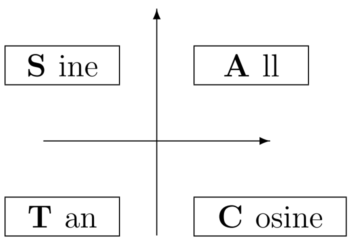
$$\sin\theta \equiv \frac{y}{h};$$

$$\cos\theta \equiv \frac{x}{h};$$

$$\tan\theta \equiv \frac{y}{x} \equiv \frac{\sin\theta}{\cos\theta}.$$

Clearly these reduce to the original definitions in the case when θ is a positive acute angle. Trigonometric functions can also be called **"trigonometric ratios"**.

(iii) It is useful to indicate diagramatically which of the three basic trigonometric functions have positive values in the various quadrants.



(iv) Three other trigonometric functions are sometimes used and are defined as the reciprocals of the three basic functions as follows:

"Secant"

$$\sec\theta \equiv \frac{1}{\cos\theta};$$

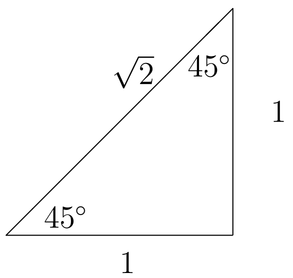
"Cosecant"

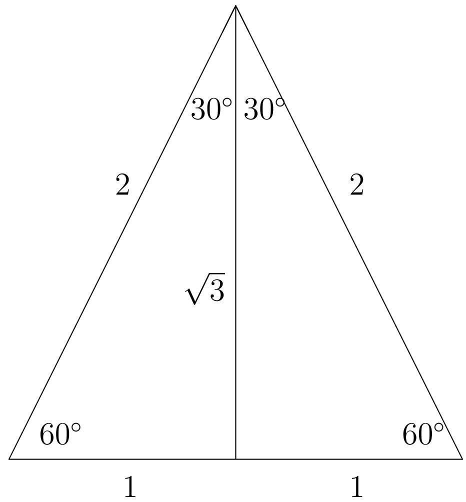
$$\mathrm{cosec}\theta \equiv \frac{1}{\sin\theta};$$

“Cotangent”

$$\cot\theta \equiv \frac{1}{\tan\theta}.$$

(v) The values of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the particluar angles 30°, 45° and 60° are easily obtained without calculator from the following diagrams:





The diagrams show that

(a) $\sin 45^\circ = \frac{1}{\sqrt{2}}$; (b) $\cos 45^\circ = \frac{1}{\sqrt{2}}$; (c) $\tan 45^\circ = 1$;

(d) $\sin 30^\circ = \frac{1}{2}$; (e) $\cos 30^\circ = \frac{\sqrt{3}}{2}$ (f) $\tan 30^\circ = \frac{1}{\sqrt{3}}$;

(g) $\sin 60^\circ = \frac{\sqrt{3}}{2}$; (h) $\cos 60^\circ = \frac{1}{2}$; (i) $\tan 60^\circ = \sqrt{3}$.

3.1.4 EXERCISES

1. Express each of the following angles as a multiple of π
   (a) 65°; (b) 105°; (c) 72°; (d) 252°;
   (e) 20°; (f) -160°; (g) 9°; (h) 279°.
2. On a circle of radius 24cms., find the length of arc which subtends an angle at the centre of
   (a) $\frac{2}{3}$ radians.; (b) $\frac{3\pi}{5}$ radians.;
   (c) 75°; (d) 130°.
3. A wheel is turning at the rate of 48 revolutions per minute. Express this angular speed in
   (a) revolutions per second; (b) radians per minute; (c) radians per second.

4. A wheel, 4 metres in diameter, is rotating at 80 revolutions per minute. Determine the distance, in metres, travelled in one second by a point on the rim.
5. A chord AB of a circle, radius 5cms., subtends a right-angle at the centre of the circle. Calculate, correct to two places of decimals, the areas of the two segments into which AB divides the circle.
6. If $\tan\theta$ is positive and $\cos\theta = -\frac{4}{5}$, what is the value of $\sin\theta$?
7. Determine the length of the chord of a circle, radius 20cms., subtending an angle of $150°$ at the centre.
8. A ladder leans against the side of a vertical building with its foot 4 metres from the building. If the ladder is inclined at $70°$ to the ground, how far from the ground is the top of the ladder and how long is the ladder ?

3.1.5 ANSWERS TO EXERCISES

1. (a) $\frac{13\pi}{36}$; (b) $\frac{7\pi}{12}$; (c) $\frac{2\pi}{5}$; (d) $\frac{7\pi}{5}$; (e) $\frac{\pi}{9}$; (f) -$\frac{8\pi}{9}$; (g) $\frac{\pi}{20}$; (h) $\frac{31\pi}{20}$.
2. (a) 16 cms.; (b) $\frac{72\pi}{5}$ cms.; (c) 10π cms.; (d) $\frac{52\pi}{3}$ cms.
3. (a) $\frac{4}{5}$ revs. per sec.; (b) 96π rads. per min.; (c) $\frac{8\pi}{5}$ rads. per sec.
4. $\frac{16\pi}{3}$ metres.
5. 7.13 square cms. and 71.41 square cms.
6. $\sin\theta = -\frac{3}{5}$.
7. The chord has a length of 38.6cms. approximately.
8. The top of ladder is 11 metres from the ground and the length of the ladder is 11.7 metres.