"JUST THE MATHS"

UNIT NUMBER

2.4

SERIES 4
(Further convergence and divergence)

by

A.J.Hobson

UNIT 2.4 - SERIES 4- FURTHER CONVERGENCE AND DIVERGENCE

2.4.1 SERIES OF POSITIVE AND NEGATIVE TERMS

Introduction

In Units 2.2 and 2.3, most of the series considered have included only positive terms. But now we shall examine the concepts of convergence and divergence in cases where negative terms are present.

We note here, for example, that the r-th Term Test encountered in Unit 2.3 may be used for series whose terms are not necessarily all positive. This is because the formula

$$u_r = S_r - S_{r-1}$$

is valid for any series.

The series cannot converge unless the partial sums S_r and S_{r-1} both tend to the same finite limit as r tends to infinity which implies that u_r tends to zero as r tends to infinity.

A particularly simple kind of series with both positive and negative terms is one whose terms are alternately positive and negative. The following test is applicable to such series:

Test 4 - The Alternating Series Test

If

$$u_1 - u_2 + u_3 - u_4 + \ . \ . \ , \quad \text{where} \quad u_r > 0,$$

is such that

$$u_r > u_{r+1} \ \ \textbf{and} \ \ u_r \to 0 \ \ \text{as} \ \ r \to \infty,$$

then the series converges.

Outline Proof:

(a) Suppose we re-group the series as

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ . \ . \ .;$$

then, it may be considered in the form

$$\sum_{r=1}^{\infty} v_r,$$

where $v_1 = u_1 - u_2, v_2 = u_3 - u_4, v_3 = u_5 - u_6, \ \ldots .$

This means that v_r is positive, so that the corresponding r-th partial sums, $S_r = v_1 + v_2 + v_3 + \ \ldots \ + v_r$, steadily increase as r increases.

(b) Alternatively, suppose we re-group the series as

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots ;$$

then, it may be considered in the form

$$u_1 - \sum_{r=1}^{\infty} w_r,$$

where $w_1 = u_2 - u_3, w_2 = u_4 - u_5, w_3 = u_6 - u_7, \ \ldots .$

In this case, each partial sum, $S_r = u_1 - (w_1 + w_2 + w_3 + \ \ldots + w_r)$ is less than u_1 since positive quantities are being subtracted from it.

(c) We conclude that the partial sums of the original series are steadily increasing but are never greater than u_1. They must therefore tend to a finite limit as r tends to infinity; that is, the series converges.

ILUSTRATION

The series

$$\sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots$$

is convergent since

$$\frac{1}{r} > \frac{1}{r+1} \ \ \textbf{and} \ \ \frac{1}{r} \rightarrow 0 \ \ \text{as} \ \ r \rightarrow \infty.$$

2.4.2 ABSOLUTE AND CONDITIONAL CONVERGENCE

In this section, a link is made between a series having both positive and negative terms and the corresponding series for which all of the terms are positive.

By making this link, we shall be able to make use of earlier tests for convergence and divergence.

DEFINITION (A)
If
$$\sum_{r=1}^{\infty} u_r$$
is a series with both positive and negative terms, it is said to be **"absolutely convergent"** if
$$\sum_{r=1}^{\infty} |u_r|$$
is convergent.

DEFINITION (B)
If
$$\sum_{r=1}^{\infty} u_r$$
is a convergent series of positive and negative terms, but
$$\sum_{r=1}^{\infty} |u_r|$$
is a divergent series, then the first of these two series is said to be **"conditionally convergent"**.

ILLUSTRATIONS

1. The series
$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$
converges absolutely since the series
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$
converges.

2. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ \ldots$$

is conditionally convergent since, although it converges (by the Alternating Series Test), the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots$$

is divergent.

Notes:

(i) It may be shown that any series of positive and negative terms which is **absolutely** convergent will also be convergent.

(ii) Any test for the convergence of a series of positive terms may be used as a test for the absolute convergence of a series of both positive and negative terms.

2.4.3 TESTS FOR ABSOLUTE CONVERGENCE

The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that $|u_r| \leq v_r$ where

$$\sum_{r=1}^{\infty} v_r$$

is a convergent series of positive terms. Then, the given series is absolutely convergent.

D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = L.$$

Then the given series is absolutely convergent if $L < 1$.

Note:

If $L > 1$, then $|u_{r+1}| > |u_r|$ for large enough values of r showing that the **numerical** values of the terms steadily increase. This implies that u_r does **not** tend to zero as r tends to infinity and, hence, by the r-th Term Test, the series diverges.

If $L = 1$, there is no conclusion.

EXAMPLES

1. Show that the series

$$\frac{1}{1\times 2} - \frac{1}{2\times 3} - \frac{1}{3\times 4} + \frac{1}{4\times 5} - \frac{1}{5\times 6} - \frac{1}{6\times 7} + \ ..$$

   is absolutely convergent.

   Solution

   The r-th term of the series is numerically equal to

$$\frac{1}{r(r+1)},$$

   which is always less than $\frac{1}{r^2}$, the r-th term of a known convergent series.

2. Show that the series

$$\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \ ..$$

   is conditionally convergent.

   Solution

   The r-th term of the series is numerically equal to

$$\frac{r}{r^2+1},$$

   which tends to zero as r tends to infinity.

   Also,

$$\frac{r}{r^2+1} > \frac{r+1}{(r+1)^2+1}$$

since this may be reduced to the true statement $r^2 + r > 1$.

Hence, by the Alternating Series Test, the series converges.

However, it is also true that

$$\frac{r}{r^2+1} > \frac{r}{r^2+r} = \frac{1}{r+1};$$

and, hence, by the Comparison Test, the series of absolute values is divergent, since

$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$

is divergent.

2.4.4 POWER SERIES

A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ . \ . \ = \sum_{r=0}^{\infty} a_r x^r \ \text{ or } \ \sum_{r=1}^{\infty} a_{r-1}x^{r-1},$$

where x is usually a variable quantity, is called a **"power Series in x with coefficients** $a_0, \ a_1, \ a_2, \ a_3, \ . \ . \ .$**"**.

Notes:

(i) In this kind of series, it is particularly useful to sum the series from $r = 0$ to infinity rather than from $r = 1$ to infinity so that the constant term at the beginning (if there is one) can be considered as the term in x^0.

But the various tests for convergence and divergence still apply in this alternative notation.

(ii) A power series will not necessarily be convergent (or divergent) for **all** values of x and it is usually required to determine the specific **range** of values of x for which the series converges. This can most frequently be done using D'Alembert's Ratio Test.

ILLUSTRATION

For the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ . \ . \ = \sum_{r=1}^{\infty} (-1)^{r-1}\frac{x^r}{r},$$

we have

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{(-1)^r x^{r+1}}{r+1}.\frac{r}{(-1)^{r-1}x^r}\right| = \left|\frac{r}{r+1}x\right|,$$

which tends to $|x|$ as r tends to infinity.

Thus, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

If $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ . \ ,$$

which converges by the Alternating Series Test; while, if $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ . \ . \ ,$$

which diverges.

The **precise** range of convergence for the given series is therefore $-1 < x \leq 1$.

2.4.5 EXERCISES

1. Show that the following alternating series are convergent:

   (a)

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ . \ . \ .;$$

   (b)

$$\frac{1}{3^2} - \frac{2}{3^3} + \frac{3}{3^4} - \frac{4}{3^5} + \ . \ .$$

2. Show that the following series are conditionally convergent:

   (a)

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ . \ . \ ;$$

(b)

$$\frac{2}{1\times 3} - \frac{3}{2\times 4} + \frac{4}{3\times 5} - \frac{5}{4\times 6} + \ \ldots$$

3. Show that the following series are absolutely convergent:

   (a)

$$\frac{3}{2} + \frac{4}{3} \times \frac{1}{2} - \frac{5}{4} \times \frac{1}{2^2} - \frac{6}{5} \times \frac{1}{2^3} + \ \ldots;$$

   (b)

$$\frac{1}{3} + \frac{1\times 2}{3\times 5} - \frac{1\times 2\times 3}{3\times 5\times 7} - \frac{1\times 2\times 3\times 4}{3\times 5\times 7\times 9} + \ ..$$

4. Obtain the precise range of values of x for which each of the following power series is convergent:

   (a)

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots;$$

   (b)

$$\frac{x}{1\times 2} + \frac{x^2}{2\times 3} + \frac{x^3}{3\times 4} + \frac{x^4}{4\times 5} + \ \ldots;$$

   (c)

$$2x + \frac{3x^2}{2^3} + \frac{4x^3}{3^3} + \frac{5x^4}{4^3} + \ \ldots;$$

   (d)

$$1 + \frac{2x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \ \ldots$$

2.4.6 ANSWERS TO EXERCISES

1. (a) Use $u_r = \frac{1}{2^{r-1}}$ (numerically);

   (b) Use $u_r = \frac{r}{3^{r+1}}$ (numerically).

2. (a) Use $u_r = \frac{1}{\sqrt{r}}$ (numerically);

   (b) Use $u_r = \frac{r+1}{r(r+2)}$ (numerically).

3. (a) Use $u_r = \frac{r+2}{r+1} \times \frac{1}{2^{r-1}}$ - (numerically);

   (b) Use $u_r = \frac{2^r (r!)^2}{(2r+1)!}$ (numerically).

4. (a) The power series converges for all values of x;

   (b) $-1 \leq x \leq 1$;

   (c) $-1 \leq x \leq 1$;

   (d) $-5 < x < 5$.

Note:
For further discussion of limiting values, see Unit 10.1