“JUST THE MATHS”

UNIT NUMBER

2.2

SERIES 2
(Binomial series)

by

A.J.Hobson

UNIT 2.2 - SERIES 2 - BINOMIAL SERIES

INTRODUCTION

In this section, we shall be concerned with the methods of expanding (multiplying out) an expression of the form

$$(A + B)^n,$$

where A and B are either mathematical expressions or numerical values, and n is a given number which need not be a positive integer. However, we shall deal first with the case when n *is* a positive integer, since there is a useful aid to memory for obtaining the result.

2.2.1 PASCAL'S TRIANGLE

Initially, we consider some simple illustrations obtainable from very elementary algebraic techniques in earlier work:

1. $(A + B)^1 \equiv$
$$A + B.$$
2. $(A + B)^2 \equiv$
$$A^2 + 2AB + B^2.$$
3. $(A + B)^3 \equiv$
$$A^3 + 3A^2B + 3AB^2 + B^3.$$
4. $(A + B)^4 \equiv$
$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4.$$

OBSERVATIONS

(i) We notice that, in each result, the expansion begins with the maximum possible power of A and ends with the maximum possible power of B.

(ii) In the sequence of terms from beginning to end, the powers of A **decrease** in steps of 1 while the powers of B **increase** in steps of 1.

(iii) The coefficients in the illustrated expansions follow the diagramatic pattern called **PASCAL'S TRIANGLE**:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

and this suggests a general pattern where each line begins and ends with the number 1 and each of the other numbers is the sum of the two numbers above it in the previous line. For example, the next line would be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

giving the result

5. $(A+B)^5 \equiv$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$

(iv) The only difference which occurs in an expansion of the form

$$(A-B)^n$$

is that the terms are alternately positive and negative. For instance,

6. $(A-B)^6 \equiv$

$$A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6.$$

2.2.2 BINOMIAL FORMULAE

In $(A+B)^n$, if n is a large positive integer, then the method of Pascal's Triangle can become very tedious. If n is *not* a positive integer, then Pascal's Triangle cannot be used anyway.

A more general method which can be applied to any value of n is the binomial formula whose proof is best obtained as an application of differential calculus and hence will not be included here.

Before stating appropriate versions of the binomial formula, we need to introduce a standard notation called a **"factorial"** by means of the following definition:

DEFINITION
If n is a positive integer, the product

$$1.2.3.4.5..........n$$

is denoted by the symbol $n!$ and is called **"n factorial"**.

Note:
This definition could not be applied to the case when $n = 0$, but it is convenient to give a meaning to 0! We define it separately by the statement

$$0! = 1$$

and the logic behind this separate definition can be made plain in the applications of calculus. There is no meaning to $n!$ when n is a negative integer.

(a) Binomial formula for $(A + B)^n$ when n is a positive integer.

It can be shown that

$$(A + B)^n \equiv A^n + na^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2 + \frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + ... + B^n.$$

Notes:

(i) This is the same as the result which would be given by Pascal's Triangle.

(ii) The last term in the expansion is really

$$\frac{n(n-1)(n-2)(n-3).......3.2.1}{n!}A^{n-n}B^n = A^0B^n = B^n.$$

(iii) The coefficient of $A^{n-r}B^r$ in the expansion is

$$\frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

and this is sometimes denoted by the symbol $\binom{n}{r}$.

(iv) A commonly used version of the result is given by

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + x^n.$$

EXAMPLES

1. Expand fully the expression $(1+2x)^3$.

   Solution

   We first note that
$$(A+B)^3 \equiv A^3 + 3A^2B + \frac{3.2}{2!}AB^2 + B^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3.$$
   If we now replace A by 1 and B by $2x$, we obtain
$$(1+2x)^3 \equiv 1 + 3(2x) + 3(2x)^2 + (2x)^3 \equiv 1 + 6x + 12x^2 + 8x^3.$$

2. Expand fully the expression $(2-x)^5$.

   Solution

   We first note that
$$(A+B)^5 \equiv A^5 + 5A^4B + \frac{5.4}{2!}A^3B^2 + \frac{5.4.3}{3!}A^2B^3 + \frac{5.4.3.2}{4!}AB^4 + B^5.$$
   That is,
$$(A+B)^5 \equiv A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$
   We now replace A by 2 and B by $-x$ to obtain
$$(2-x)^5 \equiv 2^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + (-x)^5.$$
   That is,
$$(2-x)^5 \equiv 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5.$$

(b) Binomial formula for $(A+B)^n$ when n is negative or a fraction.

It turns out that the binomial formula for a positive integer index may still be used when the index is negative or a fraction, except that the series of terms will be an **infinite** series. That is, it will not terminate.

In order to state the most commonly used version of the more general result, we use the simplified form of the binomial formula in Note (iii) of the previous section:

RESULT

If n is negative or a fraction and x lies strictly between $x=-1$ and $x=1$, it can be shown that
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

EXAMPLES

1. Expand $(1+x)^{\frac{1}{2}}$ as far as the term in x^3.

   Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + ...$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ......$$

   provided $-1 < x < 1$.

2. Expand $(2-x)^{-3}$ as far as the term in x^3 stating the values of x for which the series is valid.

   Solution

   We first convert the expression $(2-x)^{-3}$ to one in which the leading term in the bracket is 1. That is,

$$(2-x)^{-3} \equiv \left[2\left(1-\frac{x}{2}\right)\right]^{-3}$$

$$\equiv \frac{1}{8}\left(1+\left[-\frac{x}{2}\right]\right)^{-3}.$$

   The required binomial expansion is thus:

$$\frac{1}{8}\left[1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{x}{2}\right)^3+......\right].$$

   That is,

$$\frac{1}{8}\left[1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+......\right].$$

   The expansion is valid provided that $-x/2$ lies strictly between -1 and 1. This will be so when x itself lies strictly between -2 and 2.

(c) Approximate Values

The Binomial Series may be used to calculate simple approximations, as illustrated by the following example:

EXAMPLE

Evaluate $\sqrt{1.02}$ correct to five places of decimals.

Solution

Using $1.02 = 1 + 0.02$, we may say that

$$\sqrt{1.02} = (1 + 0.02)^{\frac{1}{2}}.$$

That is,

$$\sqrt{1.02} = 1 + \frac{1}{2}(0.02) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1.2}(0.02)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2.3}(0.02)^3 + \ \ldots$$

$$= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) - \ldots$$

$$= 1 + 0.01 - 0.00005 + 0.0000005 - \ \ldots$$

$$\simeq 1.010001 - 0.000050 = 1.009951$$

Hence $\sqrt{1.02} \simeq 1.00995$

2.2.3 EXERCISES

1. Expand the following, using Pascal's Triangle:

   (a)
   $$(1+x)^5;$$
   (b)
   $$(x+y)^6;$$
   (c)
   $$(x-y)^7;$$
   (d)
   $$(x-1)^8.$$

2. Use the result of question 1(a) to evaluate
   $$(1.01)^5$$
   without using a calculator.

3. Expand fully the following expressions:

   (a)
   $$(2x-1)^5;$$
   (b)
   $$\left(3+\frac{x}{2}\right)^4;$$
   (c)
   $$\left(x-\frac{2}{x}\right)^3.$$

4. Expand the following as far as the term in x^3, stating the values of x for which the expansions are valid:

   (a)
   $$(3+x)^{-1};$$

(b)

$$(1-2x)^{\frac{1}{2}};$$

(c)

$$(2+x)^{-4}.$$

5. Using the first four terms of the expansion for $(1+x)^n$, calculate an approximate value of $\sqrt{1.1}$, stating the result correct to five significant figures.

6. If x is small, show that

$$(1+x)^{-1}-(1-2x)^{\frac{1}{2}} \simeq \frac{3x^2}{2}.$$

2.2.4 ANSWERS TO EXERCISES

1. (a)

$$1+5x+10x^2+10x^3+5x^4+x^5;$$

(b)

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6;$$

(c)

$$x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7;$$

(d)

$$x^8-8x^7+28x^6-56x^5+70x^4-56x^3+28x^2-8x+1.$$

2. 1.0510100501 to ten places of decimals.

3. (a)

$$32x^5-80x^4+80x^3-40x^2+10x-1;$$

(b)

$$81+54x+\frac{27}{2}x^2+\frac{3}{2}x^3+\frac{1}{16}x^4;$$

(c)

$$x^3-6x+\frac{12}{x}-\frac{8}{x^3}.$$

4. (a)

$$\frac{1}{3}\left[1-\frac{x}{3}+\frac{x^2}{9}-\frac{x^3}{27}+......\right],$$

provided $-3 < x < 3$.

(b)

$$1-x-\frac{x^2}{2}-\frac{x^3}{2}-......,$$

provided $-\frac{1}{2} < x < \frac{1}{2}$.

(c)

$$\frac{1}{16}\left[1-2x+\frac{5x^2}{2}-\frac{5x^3}{2}+......\right],$$

provided $-2 < x < 2$.

5. 1.0488

6. Expand each bracket as far as the term in x^2.