"JUST THE MATHS"

UNIT NUMBER

2.1

SERIES 1
(Elementary progressions and series)

by

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UNIT 2.1 - SERIES 1 - ELEMENTARY PROGRESSIONS AND SERIES

2.1.1 ARITHMETIC PROGRESSIONS

The **"sequence"** of numbers,

$$a, a + d, a + 2d, a + 3d, ...$$

is said to form an **"arithmetic progression"**.

The symbol a represents the **"first term"**, the symbol d represents the **"common difference"** and the **"n-th term"** is given by the expression

$$a + (n - 1)d$$

EXAMPLES

1. Determine the n-th term of the arithmetic progression 15, 12, 9, 6,...

   Solution

   The n-th term is
   $$15 + (n - 1)(-3) = 18 - 3n.$$

2. Determine the n-th term of the arithmetic progression
   $$8, 8.125, 8.25, 8.375, 8.5, ...$$

   Solution

   The n-th term is
   $$8 + (n - 1)(0.125) = 7.875 + 0.125n.$$

3. The 13th term of an arithmetic progression is 10 and the 25th term is 20; calculate
   (a) the common difference;
   (b) the first term;
   (c) the 17th term.

   Solution

   Letting a be the first term and d be the common difference, we have
   $$a + 12d = 10$$
   and
   $$a + 24d = 20.$$

   (a) Subtracting the first of these from the second gives $12d = 10$, so that the common difference $d = \frac{10}{12} = \frac{5}{6} \simeq 0.83$

(b) Substituting into the first of the relationships between a and d gives

$$a + 12 \times \frac{5}{6} = 10.$$

That is,

$$a + 10 = 10.$$

Hence, $a = 0$.

(c) The 17th term is

$$0 + 16 \times \frac{5}{6} = \frac{80}{6} = \frac{40}{3} \simeq 13.3$$

2.1.2 ARITHMETIC SERIES

If the terms of an arithmetic progression are added together, we obtain what is called an **"arithmetic series"**. The total sum of the first n terms of such a series can be denoted by S_n so that

$$S_n = a + [a + d] + [a + 2d] + ... + [a + (n-2)d] + [a + (n-1)d];$$

but this is not the most practical way of evaluating the sum of the n terms, especially when n is a very large number.

A trick which provides us with a more convenient formula for S_n is to write down the existing formula **backwards**. That is,

$$S_n = [a + (n-1)d] + [a + (n-2)d] + ... + [a + 2d] + [a + d] + a.$$

Adding the two statements now gives

$$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + ... + [2a + (n-1)d] + [2a + (n-1)d],$$

where, on the right hand side, there are n repetitions of the same expression.

Hence,

$$2S_n = n[2a + (n-1)d]$$

or

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

This version of the formula is suitable if we know the values of a, n and d; but an alternative version can be used if we know only the first term, the last term and the number of terms. In this case,

$$S_n = \frac{n}{2}[\text{FIRST} \quad + \quad \text{LAST}]$$

which is simply n times the average of the first and last terms.

EXAMPLES

1. Determine the sum of the natural numbers from 1 to 100.
   Solution
   The sum is given by
   $$\frac{100}{2} \times [1 + 100] = 5050.$$
2. How many terms of the arithmetic series
   $$10 + 12 + 14 + ...$$
   must be taken so that the sum of the series is 252 ?
   Solution
   The first term is clearly 10 and the common difference is 2.
   Hence, letting n be the number of terms, we require that
   $$252 = \frac{n}{2}[20 + (n - 1) \times 2].$$
   That is,
   $$252 = \frac{n}{2}[2n + 18] = n(n + 9).$$
   By trial and error, $n = 12$ will balance this equation; but it is more conclusive to obtain n as the solution to the quadratic equation
   $$n^2 + 9n - 252 = 0$$
   or
   $$(n - 12)(n + 21) = 0,$$
   which gives $n = 12$ only, since the negative value $n = -21$ may be ignored.
3. A contractor agrees to sink a well 250 metres deep at a cost of £2.70 for the first metre, £2.85 for the second metre and an extra 15p for each additional metre. Find the cost of the last metre and the total cost.
   Solution
   In this problem we are dealing with an arithmetic series of 250 terms whose first term is 2.70 and whose common difference is 0.15. The cost of the last metre is the 250-th term of the series and therefore
   £$[2.70 + 249 \times 0.15] =$ £40.05
   The total cost will be
   £$\frac{250}{2} \times [2.70 + 40.05] =$ £5343.75

2.1.3 GEOMETRIC PROGRESSIONS

The sequence of numbers

$$a, ar, ar^2, ar^3, ...$$

is said to form a "geometric progression".

The symbol a represents the **"first term"**, the symbol r represents the **"common ratio"** and the **"n-th term"** is given by the expression

$$ar^{n-1}.$$

EXAMPLES

1. Determine the n-th term of the geometric progression

   $$3, -12, 48, -192, ...$$

   Solution

   The progression has n-th term

   $$3(-4)^{n-1}$$

   which will always be positive when n is an odd number and negative when n is an even number.

2. Determine the seventh term of the geometric progression

   $$3, 6, 12, 24, ...$$

   Solution

   The seventh term is

   $$3(2^6) = 192.$$

3. The third term of a geometric progression is 4.5 and the ninth term is 16.2. Determine the common ratio.

   Solution

   Firstly, we have

   $$ar^2 = 4.5$$

   and

   $$ar^8 = 16.2$$

   Dividing the second of these by the first gives

   $$\frac{ar^8}{ar^2} = \frac{16.2}{4.5}.$$

Therefore,

$$r^6 = 3.6$$

and so

$$r \simeq 1.238$$

4. The expenses of a company are £200,000 a year. It is decided that each year they shall be reduced by 5% of those for the preceding year.
   What will be the expenses during the fourth year, the first reduction taking place at the end of the first year ?

   Solution
   In this problem, we use a geometric progression with first term 200,000 and common ratio 0.95.

   The expenses during the fourth year will thus be the fourth term of the progression; that is, £200,000 $\times(0.95)^3$ = £171475.

2.1.4 GEOMETRIC SERIES

If the terms of a geometric progression are added together, we obtain what is called a **"geometric series"** . The total sum of a geometric series with n terms may be denoted by S_n so that

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

but, as with arithmetic series, this is not the most practical formula for evaluating S_n.

This time, a trick to establish a convenient formula for S_n is to write down both S_n and rS_n, the latter giving

$$rS_n = \qquad ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n.$$

Subtracting the second formula from the first gives

$$S_n - rS_n = a - ar^n,$$

so that

$$S_n = \frac{a(1-r^n)}{1-r}.$$

This is the version of the formula most commonly used since, in many practical applications, r will be less than one; but, for examples in which r is greater than one, it may be better to use the alternative version, namely

$$S_n = \frac{a(r^n-1)}{r-1}.$$

In fact, either version may be used whatever the value of r is.

EXAMPLES

1. Determine the sum of the geometric series

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

   Solution

   The sum is given by

$$S_6 = \frac{4(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{4(1 - 0.0156)}{0.5} \simeq 7.875$$

2. A sum of money £C is invested for n years at an interest of $100r\%$, compounded annually. What will be the total interest earned by the end of the n-th year ?

   Solution

   At the end of year 1, the interest earned will be Cr.
   At the end of year 2, the interest earned will be $(C + Cr)r = Cr(1 + r)$.
   At the end of year 3, the interest earned will be $C(1 + r)r + C(1 + r)r^2 = Cr(1 + r)^2$.

   This pattern reveals that,
   at the end of year n, the interest earned will be $Cr(1 + r)^{n-1}$.

   Thus the total interest earned by the end of year n will be

$$Cr + Cr(1 + r) + Cr(1 + r)^2 + ... + Cr(1 + r)^{n-1},$$

   which is a geometric series of n terms with first term Cr and common ratio $1 + r$. Its sum is therefore

$$\frac{Cr((1 + r)^n - 1)}{r} = C((1 + r)^n - 1).$$

   Note:

   The same result can be obtained using only a geometric progression as follows:

   At the end of year 1, the total amount will be $C + Cr = C(1 + r)$.
   At the end of year 2, the total amount will be $C(1 + r) + C(1 + r)r = C(1 + r)^2$.
   At the end of year 3, the total amount will be $C(1 + r)^2 + C(1 + r)^2 r = C(1 + r)^3$.
   At the end of year n, the total amount will be $C(1 + r)^n$.

   Thus the total interest earned will be $C(1 + r)^n - C = C((1 + r)^n - 1)$ as before.

The sum to infinity of a geometric series.

In a geometric series with n terms, suppose that the value of the common ratio, r, is numerically less than 1. Then the higher the value of n, the smaller the numerical value of r^n, to the extent that, as n approaches infinity, r^n approaches zero.

We conclude that, although it not possible to reach the end of a geometric series which has an infinite number of terms, its sum to infinity may be given by

$$S_\infty = \frac{a}{1-r}.$$

EXAMPLES

1. Determine the sum to infinity of the geometric series

$$5 - 1 + \frac{1}{5} - \ . \ . \ .$$

Solution

The sum to infinity is

$$\frac{5}{1+\frac{1}{5}} = \frac{25}{6} \simeq 4.17$$

2. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If, in a certain year, its output was £25,000, what could be reckoned as its total future output ?

Solution

The total output, in pounds, for subsequent years will be given by

$$25000 \times 0.75 + 25000 \times (0.75)^2 + 25000 \times (0.75)^3 + ... = \frac{25000 \times 0.75}{1-0.75} = 75000.$$

2.1.5 MORE GENERAL PROGRESSIONS AND SERIES

Introduction

Not all progressions and series encountered in mathematics are either arithmetic or geometric. For instance:

$$1^2, 2^2, 3^2, 4^2, ..., n^2$$

has a clearly defined pattern but is not arithmetic or geometric.

An **arbitrary** progression of n numbers which conform to some regular pattern is often denoted by

$$u_1, u_2, u_3, u_4, \ldots\ldots, u_n,$$

and it may or may not be possible to find a simple formula for the sum S_n.

The Sigma Notation (Σ).

If the general term of a series with n terms is known, then the complete series can be written down in short notation as indicated by the following illustrations:

1.

$$a + (a+d) + (a+2d) + \ldots\ldots + (a + [n-1]d) = \sum_{r=1}^{n} (a + [r-1]d).$$

2.

$$a + ar + ar^2 + \ldots\ldots ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

3.

$$1^2 + 2^2 + 3^2 + \ldots\ldots n^2 = \sum_{r=1}^{n} r^2.$$

4.

$$-1^3 + 2^3 - 3^3 + 4^3 + \ldots\ldots(-1)^n n^3 = \sum_{r=1}^{n} (-1)^r r^3.$$

Notes:

(i) It is sometimes more convenient to count the terms of a series from zero rather than 1. For example:

$$a + (a+d) + (a+2d) + \ldots\ldots a + [n-1]d = \sum_{r=0}^{n-1} (a + rd)$$

and

$$a + ar + ar^2 + ar^3 + ...ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

In general, for a series with n terms starting at u_0,

$$u_0 + u_1 + u_2 + u_3 + ... + u_{n-1} = \sum_{r=0}^{n-1} u_r.$$

(ii) We may also use the sigma notation for **"infinite series"** such as those we encountered in the sum to infinity of a geometric series. For example

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \quad \text{or} \quad \sum_{r=0}^{\infty} \frac{1}{3^r}.$$

STANDARD RESULTS

It may be shown that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

Outline Proofs:

The first of these is simply the formula for the sum of an arithmetic series with first term 1 and last term n.

The second is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

The third is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1.$$

EXAMPLE

Determine the sum to n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + \ldots$$

Solution

The series is

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r.$$

Using the three standard results, the summation becomes

$$\left[\frac{1}{2}n(n+1)\right]^2 + 3\left[\frac{1}{6}n(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{4}n(n+1)[n(n+1) + 4n + 2 + 4] = \frac{1}{4}n(n+1)[n^2 + 5n + 6].$$

This simplifies to

$$\frac{1}{4}n(n+1)(n+2)(n+3).$$

2.1.6 EXERCISES

1. Write down the next two terms and also the n-th term of the following sequences of numbers which are either arithmetic progressions or geometric progressions:
   (a) 40, 29, 18, 7, ...;
   (b) $\frac{13}{3}$, $\frac{17}{3}$, 7, ...;
   (c) 5, 15, 45, ...;
   (d) 10, 9.2, 8.4, ...;
   (e) 81, -54, 36, ...;
   (f) $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{16}$, ...
2. The third term of an arithmetic series is 34 and the 17th term is -8. Find the sum of the first 20 terms.
3. For the geometric series 1 + 1.2 + 1.44 + ..., find the 6th term and the sum of the first 10 terms.
4. A parent places in a savings bank £25 on his son's first birthday, £50 on his second, £75 on his third and so on, increasing the amount by £25 on each birthday. How much will be saved up (apart from any accrued interest) when the boy reaches his 16th birthday if the final amount is added on this day ?
5. Every year, a gardner takes 4 runners from each of his one year old strawberry plants in order to form 4 additional plants. If he starts with 5 plants, how many new plants will he take at the end of the 6th year and what will then be his total number of plants ?
6. A superball is dropped from a height of 10m. At each rebound, it rises to a height which is 90% of the height from which it has just fallen. What is the total distance through which the ball will have moved before it finally comes to rest ?
7. Express the series
$$\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\frac{8}{9}+......n\text{terms}$$
in both of the forms
$$\sum_{r=1}^{n} u_r \quad \text{and} \quad \sum_{r=0}^{n-1} u_r.$$
Hint:
Find the pattern in the numerators and denominators separately

8. Determine the sum to n terms of the series

$$1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \ldots$$

2.1.7 ANSWERS TO EXERCISES

1. (a) -4, -15; $51 - 11n$;
   (b) $8\frac{1}{3}$, $9\frac{2}{3}$; $\frac{1}{3}(9 + 4n)$;
   (c) 135, 405; $5(3)^{n-1}$;

   (d) 7.6, 6.8; $10.8 - 0.8n$;
   (e) -24, 16; $(-1)^{n-1}2^{n-1}3^{5-n}$;
   (f) $\frac{9}{64}$, $\frac{27}{256}$; $\frac{3^{n-2}}{4^{n-1}}$.
2. $a = 40$, $d = -3$, $S_{20} = 230$.
3. 6th term $= (1.2)^5 \simeq 2.488$ and $S_{10} \simeq 25.96$
4. £3400.
5. Number at year 6 $= 5 \times 4^6 = 20480$; Total $= 27305$.
6. Total distance $= 10 + \frac{2\times10\times0.9}{1-0.9} = 190$m.
7.
$$\sum_{r=1}^{n} \frac{2r}{2r+1} \quad \text{and} \quad \sum_{r=0}^{n-1} \frac{2(r+1)}{2r+3}.$$
8.
$$\frac{1}{6}n(n+1)(4n+5).$$