"JUST THE MATHS"

UNIT NUMBER

17.8

NUMERICAL MATHEMATICS 8
(Numerical solution)
of
(ordinary differential equations (C))

by

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UNIT 17.8 - NUMERICAL MATHEMATICS 8

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (C)

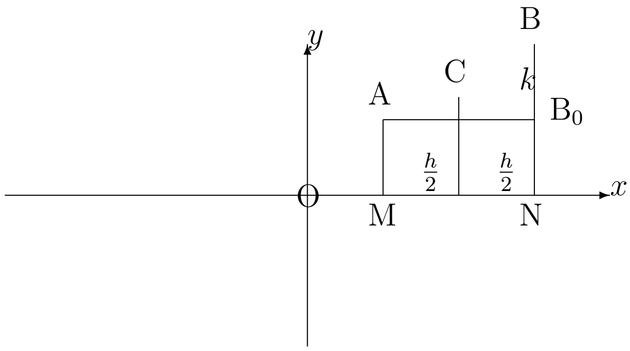
17.8.1 RUNGE'S METHOD

We solve, approximately, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),$$

subject to the condition that $y = y_0$ when $x = x_0$.

Consider the **graph** of the solution, passing through the two points, $\mathrm{A}(x_0, y_0)$ and $\mathrm{B}(x_0 + h, y_0 + k)$.



We can say that

$$\int_{x_0}^{x_0+h} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

That is,

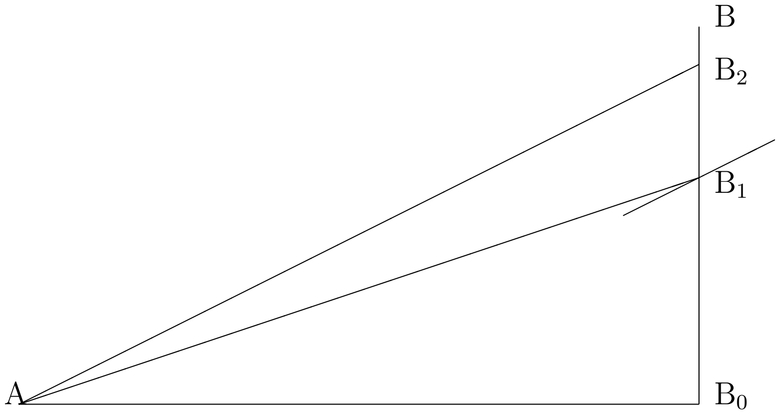
$$y_\mathrm{B} - y_\mathrm{A} = \int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

Suppose that C is the intersection with the curve of the perpendicular bisector of MN. Then, by Simpson's Rule (See Unit 17.3),

$$\int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x = \frac{h/2}{3}\left[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}\right].$$

(i) The value of $f(\mathrm{A})$
This is already given, namely, $f(x_0, y_0)$.

(ii) The Value of $f(\mathrm{B})$



In the diagram, if the tangent at A meets $\mathrm{B_0B}$ in $\mathrm{B_1}$, then the gradient at A is given by

$$\frac{\mathrm{B_1B_0}}{\mathrm{AB_0}} = f(x_0, y_0).$$

Therefore,

$$\mathrm{B_1B_0} = \mathrm{AB_0} f(x_0, y_0) = hf(x_0, y_0).$$

Calling this value k_1, as an initial approximation to k, we have

$$k_1 = hf(x_0, y_0).$$

As a rough approximation to the gradient of the solution curve passing through B, we now take the gradient of the solution curve passing through $\mathrm{B_1}$. Its value is

$$f(x_0 + h, y_0 + k_1).$$

To find a better approximation, we assume that a straight line of gradient $f(x_0 + h, y_0 + k_1)$, drawn at A, meets $\mathrm{B_0B}$ in $\mathrm{B_2}$, a point nearer to B than $\mathrm{B_1}$.

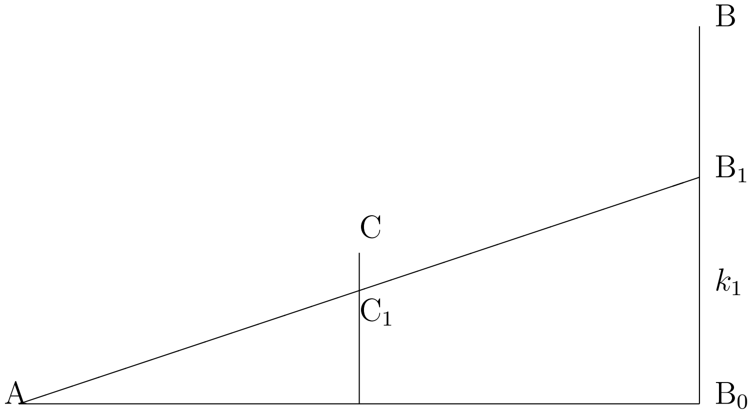
Letting $\mathrm{B_0B_2} = k_2$, we have

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

The co-ordinates of B_2 are (x_0+h, y_0+k_2) and the gradient of the solution curve through B_2 is taken as a closer approximation than before to the gradient of the solution curve through B. Its value is

$$f(x_0 + h, y_0 + k_2).$$

(iii) The Value of f(C)



Let C_1 be the intersection of the ordinate through C and the tangent at A. Then C_1 is the point,

$$\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

and the gradient at C_1 of the solution curve through C_1 is

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

We take this to be an approximation to the gradient at C for the arc, AB.

We saw earlier that

$$y_{\rm B} - y_{\rm A} = \int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x.$$

Therefore,

$$y_{\rm B} - y_{\rm A} = \frac{h}{6}[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}].$$

That is,

$$y = y_0 + \frac{h}{6}\left[f(x_0, y_0) + f(x_0 + h, y_0 + k_2) + 4f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\right].$$

PRACTICAL LAYOUT
If
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$
and $y = y_0$ when $x = x_0$, then the value of y when $x = x_0 + h$ is determined by the following sequence of calculations:

1. $k_1 = hf(x_0, y_0)$.
2. $k_2 = hf(x_0 + h, y_0 + k_1)$.
3. $k_3 = hf(x_0 + h, y_0 + k_2)$.
4. $k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.
5. $k = \frac{1}{6}(k_1 + k_3 + 4k_4)$.
6. $y \simeq y_0 + k$.

EXAMPLE

Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 3y$$
at $x = 0.1$, given that $y = 1$ when $x = 0$.

Solution

We use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

1. $k_1 = 0.1(5 - 3) = 0.2$
2. $k_2 = 0.1(5 - 3[1.2]) = 0.14$
3. $k_3 = 0.1(5 - 3[1.14]) = 0.158$
4. $k_4 = 0.1(5 - 3[1.1]) = 0.17$
5. $k = \frac{1}{6}(0.2 + 0.158 + 4[0.17]) = 0.173$
6. $y \simeq 1.173$ at $x = 0.1$

Note:
It can be shown that the error in the result is of the order h^5; that is, the error is equivalent to some constant multiplied by h^5.

17.8.2 EXERCISES

1. Use Runge's Method to solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$$
at $x = 0.3$, given that $y = 0$ when $x = 0$.

Work to four places of decimals throughout.

2. Use Runge's Method to solve the differential equation

$$\frac{dy}{dx} = \frac{y}{y+x}$$

at $x = 1.1$, given that $y(1) = 1$.

Work to three places of decimals throughout.

3. Use Runge's Method with **successive** increments of $h = 0.1$ to find the solution at $x = 0.5$ of the differential equation

$$\frac{dy}{dx} = xy,$$

given that $y(0) = 1$. Work to four decimal places throughout.

Compare your results with those given by the exact solution

$$y = e^{\frac{1}{2}x^2}.$$

4. Use Runge's Method with $h = 0.2$ to determine the solution at $x = 1$ of the differential equation,

$$\frac{dy}{dx} = y + e^{-x},$$

given that $y(0) = 0$. Work to four decimal places throughout.

17.8.3 ANSWERS TO EXERCISES

1.

$$y(0.3) \simeq 0.0454$$

2.

$$y(1.1) \simeq 1.049$$

3.

$$y(0.5) \simeq 1.3318$$

4.

$$y(1) \simeq 1.1752$$