"JUST THE MATHS"

UNIT NUMBER

17.7

NUMERICAL MATHEMATICS 7
(Numerical solution)
of
(ordinary differential equations (B))

by

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UNIT 17.7 - NUMERICAL MATHEMATICS 7

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (B)

17.7.1 PICARD'S METHOD

This method of solving a differential equation approximately is one of successive approximation; that is, it is an **iterative** method in which the numerical results become more and more accurate, the more times it is used.

An approximate value of y (taken, at first, to be a constant) is substituted into the right hand side of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y).$$

The equation is then integrated with respect to x giving y in terms of x as a second approximation, into which given numerical values are substituted and the result rounded off to an assigned number of decimal places or significant figures.

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

A hint on notation
Imagine, for example, that we wished to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

given that $y = y_0 = 7$ when $x = x_0 = 2$.

This ofcourse can be solved exactly to give

$$y = x^3 + C,$$

which requires that

$$7 = 2^3 + C.$$

Hence,

$$y - 7 = x^3 - 2^3;$$

or, in more general terms

$$y - y_0 = x^3 - x_0^3.$$

Thus,

$$\int_{y_0}^{y} \mathrm{d}y = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

In other words,

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

The rule, in future, therefore, will be to integrate both sides of the given differential equation with respect to x, from x_0 to x.

EXAMPLES

1. Given that

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2,$$

   and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

   Solution

   To begin the solution, we proceed as follows:

   $$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

   where $x_0 = 0$.

   Hence,

   $$y - y_0 = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

   where $y_0 = 0$.

   That is,

   $$y = \int_{0}^{x} (x + y^2) \mathrm{d}x.$$

   (a) First Iteration

   We do not know y in terms of x yet, so we replace y by the constant value y_0 in the function to be integrated.

   The result of the first iteration is thus given, at $x = 0.3$, by

   $$y_1 = \int_{0}^{x} x \mathrm{d}x = \frac{x^2}{2} \simeq 0.0450$$

(b) Second Iteration
Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_1^2 = x + \frac{x^4}{4}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20}.$$

The result of the second iteration is thus given by

$$y_2 = \frac{x^2}{2} + \frac{x^5}{20} \simeq 0.0451$$

at $x = 0.3$.

(c) Third Iteration
Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_2^2$$
$$= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

The result of the third iteration is thus given by

$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \simeq 0.0451 \quad at \quad x = 0.3$$

Hence, $y = 0.0451$, correct to four decimal places, at $x = 0.3$.

2. If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y}{x}$$

and $y = 2$ when $x = 1$, perform three iterations of Picard's method to estimate a value for y when $x = 1.2$. Work to four places of decimals throughout and state how accurate is the result of the third iteration.

Solution

(a) First Iteration

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $x_0 = 1$.

That is,

$$y - y_0 = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $y_0 = 2$.

Hence,

$$y - 2 = \int_{1}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x.$$

Replacing y by $y_0 = 2$ in the function being integrated, we have

$$y - 2 = \int_{1}^{x} \left(2 - \frac{2}{x}\right) \mathrm{d}x.$$

Therefore,

$$\begin{aligned} y &= 2 + [2x - 2\ln x]_1^x \\ &= 2 + 2x - 2\ln x - 2 + 2\ln 1 = 2(x - \ln x). \end{aligned}$$

The result of the first iteration is thus given by

$$y_1 = 2(x - \ln x) \simeq 2.0354,$$

when $x = 1.2$.

(b) Second Iteration

In this case we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_1}{x} = 2 - \frac{2(x - \ln x)}{x} = \frac{2\ln x}{x}.$$

Hence,

$$\int_{1}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{1}^{x} \frac{2lnx}{x}\mathrm{d}x.$$

That is,

$$y - 2 = \left[(\ln x)^2\right]_1^x = (\ln x)^2.$$

The result of the second iteration is thus given by

$$y_2 = 2 + (\ln x)^2 \simeq 2.0332,$$

when $x = 1.2$.

(c) Third Iteration

Finally, we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_2}{x} = 2 - \frac{2}{x} - \frac{(\ln x)^2}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \left[2 - \frac{2}{x} - \frac{(lnx)^2}{x}\right] \mathrm{d}x.$$

That is,

$$y - 2 = \left[2x - 2\ln x - \frac{(\ln x)^3}{3}\right]_1^x$$

$$= 2x - 2\ln x - \frac{(\ln x)^3}{3} - 2.$$

The result of the third iteration is thus given by

$$y_3 = 2x - 2\ln x - \frac{(\ln x)^3}{3} \simeq 2.0293,$$

when $x = 1.2$.

The results of the last two iterations are identical when rounded off to two places of decimals, namely 2.03. Hence, the accuracy of the third iteration is two decimal place accuracy.

17.7.2 EXERCISES

1. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^x$$

at $x = 1$, correct to two significant figures, given that $y = 0$ when $x = 0$.

2. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2}$$

at $x = 0.5$, correct to two decimal places, given that $y = 1$ when $x = 0$.

3. Given the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - xy,$$

where $y(0) = 0$, use Picard's method to obtain y as a series of powers of x which will give two decimal place accuracy in the interval $0 \leq x \leq 1$.

What is the solution when $x = 1$?

17.7.3 ANSWERS TO EXERCISES

1.

$$y(1) \simeq 2.7$$

2.

$$y(0.5) \simeq 1.33$$

3.

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - ......$$

$$y(1) \simeq 0.72$$