"JUST THE MATHS"

UNIT NUMBER

17.6

NUMERICAL MATHEMATICS 6
(Numerical solution)
of
(ordinary differential equations (A))

by

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UNIT 17.6 - NUMERICAL MATHEMATICS 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (A)

17.6.1 EULER'S UNMODIFIED METHOD

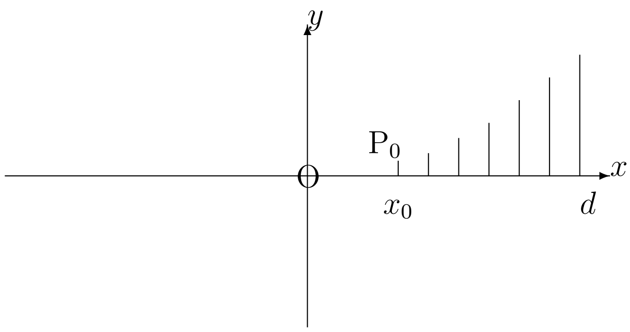
Every first order ordinary differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y);$$

and, if it is given that $y = y_0$ when $x = x_0$, then the solution for y in terms of x represents some curve through the point $\mathrm{P}_0(x_0, y_0)$.

Suppose that we wish to find the solution for y at $x = d$, where $d > x_0$.

We sub-divide the interval from $x = x_0$ to $x = d$ into n equal parts of width, δx.



Letting $x_1, x_2, x_3, ...$ be the points of subdivision, we have

$$x_1 = x_0 + \delta x,$$

$$x_2 = x_0 + 2\delta x,$$

$$x_3 = x_0 + 3\delta x,$$

$$...,$$

$$...,$$

$$d = x_n = x_0 + n\delta x.$$

If $y_1, y_2, y_3, ...$ are the y co-ordinates of $x_1, x_2, x_3, ...$, we are required to find y_n.

From elementary calculus, the increase in y, when x increases by δx, is given approximately by $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; and since, in our case, $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, we have

$$y_1 = y_0 + f(x_0, y_0)\delta x,$$

$$y_2 = y_1 + f(x_1, y_1)\delta x,$$

$$y_3 = y_2 + f(x_2, y_2)\delta x,$$

$$...,$$

$$...,$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\delta x,$$

each stage using the previously calculated y value.

Note:
The method will be the same if $d < x_0$, except that δx will be negative.

In general, each intermediate value of y is given by the formula

$$y_{i+1} = y_i + f(x_i, y_i)\delta x.$$

EXAMPLE

Use Euler's method with 5 sub-intervals to continue, to $x = 0.5$, the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

Solution

i	x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + f(x_i, y_i)\delta x$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.061106	0.4244424	1.1035524
5	0.5	1.1035524	-	-

Accuracy
The differential equation in the above example is simple to solve by an elementary method,

such as separation of the variables. It is therefore useful to compare the exact result so obtained with the approximation which comes from Eulers' method.

$$\int \frac{\mathrm{d}y}{y} = \int x\mathrm{d}x.$$

Therefore

$$\ln y = \frac{x^2}{2} + C;$$

that is,

$$y = Ae^{\frac{x^2}{2}}.$$

At $x = 0$, we are told that $y = 1$ and, hence, $A = 1$, giving

$$y = e^{\frac{x^2}{2}}.$$

But a table of values of x against y in the previous interval reveals the following:

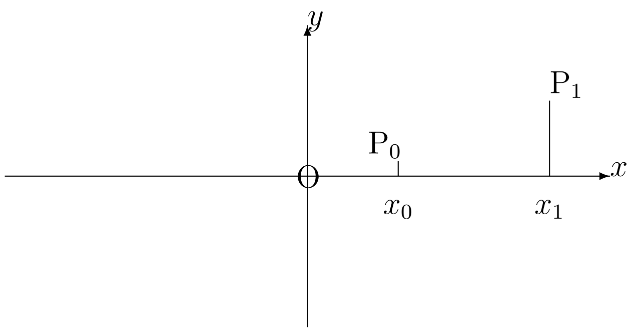
x	$e^{\frac{x^2}{2}}$
0	1
0.1	1.00501
0.2	1.0202
0.3	1.04603
0.4	1.08329
0.5	1.13315

There is thus an error in our approximate value of 0.0296, which is about 2.6%. Attempts to determine y for values of x which are greater than 0.5 would result in a very rapid growth of error.

17.6.2 EULER'S MODIFIED METHOD

In the previous method, we used the gradient to the solution curve at the point P_0 in order to find an approximate position for the point P_1, and so on up to P_n.

But the approximation turns out to be much better if, instead, we use the **average** of the two gradients at P_0 and P_1 for which we need use only x_0, y_0 and δx in order to calculate approximately.



The gradient, m_0, at P_0, is given by

$$m_0 = f(x_0, y_0).$$

The gradient, m_1, at P_1, is given approximately by

$$m_1 = f(x_0 + \delta x, y_0 + \delta y_0),$$

where $\delta y_0 = f(x_0, y_0)\delta x$.

Note:
We cannot call $y_0 + \delta y_0$ by the name y_1, as we did with the unmodified method, because this label is now reserved for the new and **better** approximation at $x = x_0 + \delta x$.

The average gradient, between P_0 and P_1, is given by

$$m_0^* = \frac{1}{2}(m_0 + m_1).$$

Hence, our approximation to y at the point P_1 is given by

$$y_1 = y_0 + m_0^* \delta x.$$

Similarly, we proceed from y_1 to y_2, and so on until we reach y_n.

In general, the intermediate values of y are given by

$$y_{i+1} = y_i + m_i^* \delta x.$$

EXAMPLE

Solve the example in the previous section using Euler's Modified method.

Solution

i	x_i	y_i	$m_i = f(x_i, y_i)$	$\delta y_i = f(x_i, y_i)\delta x$	$m_{i+1} = f(x_i + \delta x, y_i + \delta y_i)$	$m_i^* = \frac{1}{2}(m_i + m_{i+1})$	$y_{i+1} = y_i + m_i^* \delta x$
0	0	1	0	0	0.1	0.05	1.005
1	0.1	1.005	0.1005	0.0101	0.2030	0.1518	1.0202
2	0.2	1.0202	0.2040	0.0204	0.3122	0.2581	1.0460
3	0.3	1.0460	0.3138	0.0314	0.4310	0.3724	1.0832
4	0.4	1.0832	0.4333	0.0433	0.5633	0.4983	1.1330
5	0.5	1.1330	——	——	——	——	——

17.6.3 EXERCISES

1. (a) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{dy}{dx} + y = 0,$$
and that $y(0) = 1$
Compare your solution with the exact solution given by
$$y = e^{-x}.$$

   (b) Taking intervals $\delta x = 0.1$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{dy}{dx} = \frac{x^2 + y}{x}$$
and that $y(0.5) = 0.5$.
Compare your solution with the exact solution given by
$$y = x^2 + \frac{x}{2}.$$

   (c) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{dy}{dx} = y + e^{-x},$$
and that $y(0) = 0$.
Compare your solution with the exact solution given by
$$y = \sinh x.$$

(d) Given that $y(1) = 2$, use Euler's unmodified method to continue the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2},$$

to obtain values of y for values of x from $x = 1$ to $x = 1.5$, in steps of 0.1.

2. Repeat all parts of question 1 using Euler's modified method.

17.6.4 ANSWERS TO EXERCISES

1. (a) 0.33, 0.37, 11% low;
   (b) 1.45 1.50. 3% low;
   (c) 1.113, 1.175, 5% low;
   (d) 2.0000, 2.200, 2.431 2.697 3.001, 3.347.
2. (a) 0.371, 0.368, 0.8% high;
   (b) 1.495, 1.50, 0.33% low;
   (c) 1.175 , accurate to three decimal places;
   (d) 2.000, 2.216, 2.465, 2.751, 3.079, 3.452.

Note:
In questions 1(d) and 2(d), the actual values are 2.000, 2.245, 2.496, 2.784, 3.113 and 3.489, from the exact solution of the differential equation.