"JUST THE MATHS"

UNIT NUMBER

17.5

NUMERICAL MATHEMATICS 5
(Iterative methods)
for solving
(simultaneous linear equations)

by

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UNIT 17.5 - NUMERICAL MATHEMATICS 5

ITERATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS

17.5.1 INTRODUCTION

An iterative method is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy. For example, if results are required to be accurate to five places of decimals, the number of **"iterations"** (that is, stages of the method) is continued until two consecutive iterations give the same result when rounded off to that number of decimal places. It is usually enough for the calculations themselves to be carried out to **two extra** places of decimals.

A similar interpretation holds for accuracy which requires a certain number of **significant figures**.

In the work which follows, we shall discuss two standard methods of solving a set of simultaneous linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3, \end{aligned}$$

when the system is **"diagonally dominant"**, which, in this case, means that

$$\begin{aligned} |a_1| &> |b_1| + |c_1|, \\ |b_2| &> |a_2| + |c_2|, \\ |c_3| &> |a_3| + |b_3|. \end{aligned}$$

The methods would be adaptable to a different number of simultaneous equations.

17.5.2 THE GAUSS-JACOBI ITERATION

This method begins by making x the subject of the first equation, y the subject of the second equation and z the subject of the third equation.
An initial approximation such as $x_0 = 1, y_0 = 1, z_0 = 1$ is substituted on the new right-hand sides to give values $x = x_1, y = y_1$ and $z = z_1$ on the new left-hand sides.

A continuation of the process leads to the following general scheme for the results of the $(n+1)$-th iteration:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_n \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_n - b_3 y_n \qquad\right).
\end{aligned}
$$

This scheme will now be illustrated by numerical examples:

EXAMPLES

1. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$
\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12,
\end{aligned}
$$

obtaining x, y and z correct to the nearest whole number.

Solution

We have

$$
\begin{aligned}
x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n.
\end{aligned}
$$

Using

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

we obtain

$$x_1 = 0.8, y_1 = 3.0, z_1 = 2.6,$$
$$x_2 = 0.72, y_2 = 2.25, z_2 = 3.44,$$
$$x_3 = 1.038, y_3 = 1.85, z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, y = 2, z = 3,$$

when rounded to the nearest whole number.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

2. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} x + 7y - z &= 3, \\ 5x + y + z &= 9, \\ -3x + 2y + 7z &= 17, \end{aligned}$$

obtaining x, y and z correct to the nearest whole number.

Solution

This set of equations is not diagonally dominant; but they can be rewritten as

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17, \end{aligned}$$

which **is** a diagonally dominant set. We could also interchange the first two of the original equations.

We have now

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_n - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_n - 0.29y_n. \end{aligned}$$

Using

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

we obtain

$$y_1 = 0.43, x_1 = 1.4, z_1 = 2.57,$$
$$y_2 = 0.59, x_2 = 1.2, z_2 = 2.91,$$
$$y_3 = 0.67, x_3 = 1.1, z_3 = 2.78$$

This is now enough to conclude that $x = 1, y = 1, z = 3$ to the nearest whole number though, this time, they are not the exact solutions.

17.5.3 THE GAUSS-SEIDEL ITERATION

This method differs from the Gauss-Jacobi Iteration in that successive approximations are used within each step **as soon as they become available**.

It turns out that the rate of convergence of this method is usually faster than that of the Gauss-Jacobi method.

The scheme of the calculations is according to the following pattern:

$$\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad\qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_{n+1} \qquad\qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_{n+1} - b_3 y_{n+1} \qquad\qquad\right).
\end{aligned}$$

EXAMPLES

1. Use the Gauss-Seidel method to solve the simultaneous linear equations

$$\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12.
\end{aligned}$$

Solution

This time, we write:

$$\begin{aligned}
x_{n+1} &= 0.8 \qquad\qquad - 0.2y_n + 0.2z_n,\\
y_{n+1} &= 3.75 - 0.25x_{n+1} \qquad\quad - 0.5z_n,\\
z_{n+1} &= 2.4 - 0.2x_{n+1} + 0.4y_{n+1},
\end{aligned}$$

and the sequence of successive results is as follows:

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

$$x_1 = 0.8, y_1 = 3.05, z_1 = 3.46,$$

$$x_2 = 0.88, y_2 = 1.80, z_2 = 2.94,$$

$$x_3 = 1.03, y_3 = 2.02, z_3 = 3.00$$

In this particular example, the rate of convergence is about the same as for the Gauss-Jacobi method, giving $x = 1, y = 2, z = 3$ to the nearest whole number; but we would normally expect the Gauss-Seidel method to converge at a faster rate.

2. Use the Gauss Seidel method to solve the simultaneous linear equations:

$$\begin{aligned}
7y + x - z &= 3,\\
y + 5x + z &= 9,\\
2y - 3x + 7z &= 17.
\end{aligned}$$

Solution

These equations give rise to the following iterative scheme:

$$\begin{aligned}
y_{n+1} &= 0.43 - 0.14x_n \qquad\quad + 0.14z_n,\\
x_{n+1} &= 1.8 \qquad\qquad - 0.2y_{n+1} - 0.2z_n,\\
z_{n+1} &= 2.43 + 0.43x_{n+1} - 0.29y_{n+1},
\end{aligned}$$

The sequence of successive results is:

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

$$y_1 = 0.43, x_1 = 1.51, z_1 = 2.96,$$
$$y_2 = 0.63, x_2 = 1.08, z_2 = 2.71,$$
$$y_3 = 0.66, x_3 = 1.13, z_3 = 2.73$$

Once more, to the nearest whole number, the solutions are $x = 1, y = 1, z = 3$.

17.5.4 EXERCISES

1. Setting $x_0 = y_0 = z_0 = 1$ and working to three places of decimals, complete four iterations of

   (a) The Gauss-Jacobi method

   and

   (b) The Gauss-Seidel method

   for the system of simultaneous linear equations

$$\begin{aligned} 7x - y + z &= 7.3, \\ 2x - 8y - z &= -6.4, \\ x + 2y + 9z &= 13.6 \end{aligned}$$

   To how many decimal places are your results accurate ?

2. Rearrange the following equations to form a diagonally dominant system and perform the first four iterations of the Gauss-Seidel method, setting $x_0 = 1$, $y_0 = 1$ and $z_0 = 1$ and working to two places of decimals:

$$\begin{aligned} x + 5y - z &= 8, \\ -9x + 3y + 2z &= 3, \\ x + 2y + 7z &= 26. \end{aligned}$$

   Estimate the accuracy of your results and suggest the exact solutions (checking that they are valid).

3. Use an appropriate number of iterations of the Gauss-Seidel method to solve accurately, to three places of decimals, the simultaneous linear equations

$$\begin{aligned} 7x + y + z &= 5, \\ -2x + 9y + 3z &= 4, \\ x + 4y + 8z &= 3. \end{aligned}$$

17.5.5 ANSWERS TO EXERCISES

1. (a) $x_4 \simeq 1.014$, $y_4 \simeq 0.850$, $z_4 \simeq 1.199$, which are accurate to one decimal place.
   (b) $x_4 \simeq 0.999$, $y_4 \simeq 0.900$, $z_4 \simeq 1.200$, which are accurate to two decimal places.
2. On interchanging the first two equations, $x_4 \simeq 0.98$, $y_4 \simeq 2.00$, $z_4 \simeq 2.99$, which are accurate to the nearest whole number. The exact solutions are $x = 1$, $y = 2$, $z = 3$.
3. $x \simeq 0.630$, $y \simeq 0.582$, $z \simeq 0.004$.