"JUST THE MATHS"

UNIT NUMBER

17.4

NUMERICAL MATHEMATICS 4
(Further Gaussian elimination)

by

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UNIT 17.4 - NUMERICAL MATHEMATICS 4
FURTHER GAUSSIAN ELIMINATION

The **elementary** method of Gaussian Elimination, for simultaneous linear equations, was discussed in Unit 9.4. We introduce, here, a more **general** method, suitable for use with sets of equations having **decimal** coefficients.

17.4.1 GAUSSIAN ELIMINATION BY "PARTIAL PIVOTING" WITH A CHECK COLUMN

Let us first consider an example in which the coefficients are **integers**.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We may set out the solution, in the form of a **table** (rather than a **matrix**) indicating each of the **"pivot elements"** in a box as follows:

	x	y	z	constant	Σ
	$\boxed{2}$	1	1	3	7
$\frac{1}{2}$	1	-2	-1	2	0
$\frac{3}{2}$	3	-1	1	8	11
		$\boxed{\frac{-5}{2}}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{-7}{2}$
1		$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
			1	3	4

INSTRUCTIONS

(i) Divide the coefficients of x in lines 2 and 3 by the coefficient of x in line 1 and write the respective results at the side of lines 2 and 3; (that is, $\frac{1}{2}$ and $\frac{3}{2}$ in this case).

(ii) Eliminate x by subtracting $\frac{1}{2}$ times line 1 from line 2 and $\frac{3}{2}$ times line 1 from line 3.

(iii) Repeat the process starting with lines 4 and 5.

(iv) line 6 implies that $z = 3$ and by substitution back into earlier lines, we obtain the values $y = -2$ and $x = 1$.

OBSERVATIONS

Difficulties could arise if the pivot element were very small compared with the other quantities in the same column, since the errors involved in dividing by small numbers are likely to be large.

A better choice of pivot element would be the one with the **largest** numerical value in its column.

We shall consider an example, now, in which this choice of pivot is made. The working will be carried out using fractional quantities; though, in practice, decimals would normally be used instead.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 5, \\ 2x + y - z &= 1, \\ x + 3y - z &= 4. \end{aligned}$$

Solution

	x	y	z	constant	Σ
$\frac{1}{2}$	1	-1	2	5	7
	$\boxed{2}$	1	-1	1	3
$\frac{1}{2}$	1	3	-1	4	7

On eliminating x, we obtain the new table:

	y	z	constant	Σ
$\frac{-3}{5}$	$\frac{-3}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
	$\boxed{\frac{5}{2}}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Eliminating y takes us to the final table as follows:

z	constant	Σ
$\frac{11}{5}$	$\frac{33}{5}$	$\frac{44}{5}$

We conclude that
$11z = 33$ and, hence, $\boxed{z = 3}$.

Substituting into the second table (either line will do), we have
$5y - 3 = 7$ and, hence, $\boxed{y = 2}$.

Substituting into the original table (any line will do), we have
$x - 2 + 6 = 5$, so that $\boxed{x = 1}$.

Notes:

(i) In questions which involve decimal quantities stated to n decimal places, the calculations should be carried out to $n + 2$ decimal places to allow for rounding up.

(ii) A final check on accuracy in the above example is obtained by adding the original three equations together and verifying that the solution obtained also satisfies the further equation

$$4x + 3y = 10.$$

(iii) It is not essential to set out the solution in the form of separate tables (at each step) with their own headings. A continuation of the first table is acceptable.

17.4.2 EXERCISES

1. Use Gaussian Elimination by Partial Pivoting with a check column to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5, \\ 3x_1 - x_2 + 2x_3 &= 8, \\ 4x_1 - 6x_2 - 4x_3 &= -2. \end{aligned}$$

   (b)

$$\begin{aligned} 5i_1 - i_2 + 2i_3 &= 3, \\ 2i_1 + 4i_2 + i_3 &= 8, \\ i_1 + 3i_2 - 3i_3 &= 2; \end{aligned}$$

(c)

$$\begin{aligned} i_1 + 2i_2 + 3i_3 &= -4, \\ 2i_1 + 6i_2 - 3i_3 &= 33, \\ 4i_1 - 2i_2 + i_3 &= 3; \end{aligned}$$

(d)

$$\begin{aligned} 7i_1 - 4i_2 &= 12, \\ -4i_1 + 12i_2 - 6i_3 &= 0, \\ -6i_2 + 14i_3 &= 0; \end{aligned}$$

2. Use Gaussian Elimination with Partial Pivoting and a check column to solve the following sets of simultaneous linear equations:

(a)

$$\begin{aligned} 1.202x_1 - 4.371x_2 + 0.651x_3 &= 19.447, \\ -3.141x_1 + 2.243x_2 - 1.626x_3 &= -13.702, \\ 0.268x_1 - 0.876x_2 + 1.341x_3 &= 6.849; \end{aligned}$$

(b)

$$\begin{aligned} -2.381x_1 + 1.652x_2 - 1.243x_3 &= 12.337, \\ 2.151x_1 - 3.427x_2 + 3.519x_3 &= 9.212, \\ 1.882x_1 + 2.734x_2 - 1.114x_3 &= 5.735; \end{aligned}$$

17.4.3 ANSWERS TO EXERCISES

1. (a) $x_1 = -1, \quad x_2 = -3, \quad x_3 = 4;$
   (b) $i_1 = 0.5, \quad i_2 = 1.5, \quad i_3 = 1.0;$
   (c) $i_1 = 3.0, \quad i_2 = 2.5, \quad i_3 = -4.0;$
   (d) $i_1 = 2.26, \quad i_2 = 0.96, \quad i_3 = 0.41$
2. (a) $x_1 = 0.229, \quad x_2 = -4.024, \quad x_3 = 2.433;$
   (b) $x_1 = -5.753, \quad x_2 = 14.187, \quad x_3 = 19.951$