"JUST THE MATHS"

UNIT NUMBER

17.3

NUMERICAL MATHEMATICS 3
(Approximate integration (B))

by

A.J.Hobson

UNIT 17.3 - NUMERICAL MATHEMATICS 3

APPROXIMATE INTEGRATION (B)

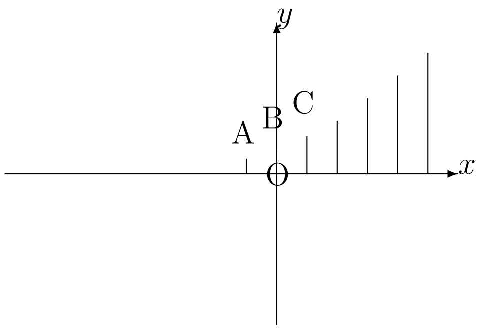
17.3.1 SIMPSON'S RULE

A better approximation to

$$\int_a^b f(x)\mathrm{d}x$$

than that provided by the Trapezoidal rule (Unit 17.2) may be obtained by usingan **even** number of narrow strips of width, h, and considering them in pairs.

To begin with, we examine a **special** case in which the first strip lies to the left of the y-axis as in the following diagram:



The arc of the curve passing through the points A$(-h, y_1)$, B$(0, y_2)$ and C(h, y_3) may be regarded as an arc of a parabola whose equation is

$$y = Lx^2 + Mx + N,$$

provided that the coefficients L, M and N satisfy the equations

$$
\begin{aligned}
y_1 &= Lh^2 - Mh + N, \\
y_2 &= N, \\
y_3 &= Lh^2 + Mh + N.
\end{aligned}
$$

Also, the area of the first pair of strips is given by

$$\text{Area} = \int_{-h}^{h} (Lx^2 + Mx + N)\ \mathrm{d}x$$

$$= \left[L\frac{x^3}{3} + M\frac{x^2}{2} + Nx\right]_{-h}^{h}$$

$$= \frac{2Lh^3}{3} + 2Nh$$

$$= \frac{h}{3}[2Lh^2 + 6N],$$

which, from the simultaneous equations earlier, gives

$$\text{Area} = \frac{h}{3}[y_1 + y_3 + 4y_2].$$

But the area of **every** pair of strips will be dependent only on the three corresponding y co-ordinates, together with the value of h.

Hence, the area of the next pair of strips will be

$$\frac{h}{3}[y_3 + y_5 + 4y_4],$$

and the area of the pair after that will be

$$\frac{h}{3}[y_5 + y_7 + 4y_6].$$

Thus, the total area is given by

$$\text{Area} = \frac{h}{3}[y_1 + y_n + 4(y_2 + y_4 + y_6 + \ldots) + 2(y_3 + y_5 + y_7 + \ldots),$$

usually interpreted as

$$\text{Area} = \frac{h}{3}[\text{First}+\text{Last}+4\times\text{The even numbered } y \text{ co} - \text{ords.}+2\times\text{The remaining } y \text{ co} - \text{ords.}]$$

or

$$\text{Area} = \frac{h}{3}[\text{F} + \text{L} + 4\text{E} + 2\text{R}]$$

This result is known as SIMPSON'S RULE.

Notes:

(i) Since the area of the pairs of strips depends only on the three corresponding y co-ordinates, together with the value of h, the Simpson's rule formula provides an approximate value of the definite integral

$$\int_a^b f(x) \, \mathrm{d}x$$

whatever the values of a and b are, as long as the curve does not cross the x-axis between $x = a$ and $x = b$.

(ii) If the curve **does** cross the x-axis between $x = a$ and $x = b$, it is necessary to consider separately the positive parts of the area above the x-axis and the negative parts below the x-axis.

(iii) The approximate evaluation, by Simpson's rule, of a definite integral should be set out in **tabular form**, as illustrated in the examples overleaf.

EXAMPLES

1. Working to a maximum of three places of decimals throughout, use Simpson's rule with ten divisions to evaluate, approximately, the definite integral

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

x_i	$y_i = e^{x_i^2}$	F & L	E	R
0	1	1		
0.1	1.010		1.010	
0.2	1.041			1.041
0.3	1.094		1.094	
0.4	1.174			1.174
0.5	1.284		1.284	
0.6	1.433			1.433
0.7	1.632		1.632	
0.8	1.896			1.896
0.9	2.248		2.248	
1.0	2.718	2.718		
	F + L $\rightarrow$	3.718	7.268	5.544
	4E $\rightarrow$	29.072	$\times 4$	$\times 2$
	2R $\rightarrow$	11.088	29.072	11.088
	(F + L) + 4E + 2R $\rightarrow$	43.878	//////	//////

Hence,

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.1}{3} \times 43.878 \simeq 1.463$$

2. Working to a maximum of three places of decimals throughout, use Simpson's rule with eight divisions between $x = -1$ and $x = 1$ and four divisions between $x = 1$ and $x = 2$ in order to evaluate, approximately, the area between the curve whose equation is

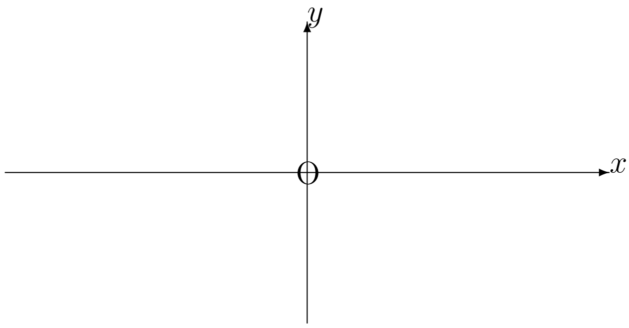
$$y = (x^2 - 1)e^{-x}$$

and the x-axis from $x = -1$ to $x = 2$.

Solution

We note that the curve crosses the x-axis when $x = -1$ and when $x = 1$, the y co-ordinates being negative in the interval between these two values of x and positive outside this interval.

Hence, we need to evaluate the negative area between $x = -1$ and $x = 1$ and the positive area between $x = 1$ and $x = 2$; then we add their numerical values together to find the total area.



(a) The Negative Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
-1	0	0		
-0.75	-0.926		-0.926	
-0.5	-1.237			-1.237
-0.25	-1.204		-1.204	
0	-1			-1
0.25	-0.730		-0.730	
0.50	-0.455			-0.455
0.75	-0.207		-0.207	
1	0	0		
	F + L $\rightarrow$	0	-2.860	-2.692
	4E $\rightarrow$	-11.440	$\times 4$	$\times 2$
	2R $\rightarrow$	-5.384	-11.440	-5.384
	(F + L) + 4E + 2R $\rightarrow$	-16.824	//////	//////

(b) The Positive Area

x_i	$y_i = (x^2-1)e^{-x}$	F & L	E	R
1	0	0		
1.25	0.161		0.161	
1.5	0.279			0.279
1.75	0.358		0.358	
2	0.406	0.406		
	F + L $\rightarrow$	0.406	0.519	0.279
	4E $\rightarrow$	2.076	$\times 4$	$\times 2$
	2R $\rightarrow$	0.558	2.076	0.558
	(F + L) + 4E + 2R $\rightarrow$	3.040	//////	//////

The total area is thus

$$\frac{0.25}{3} \times (16.824 + 3.040) \simeq 1.655$$

17.3.2 EXERCISES

Use Simpson's rule with six divisions of the x-axis to find an approximation for each of the following, working to a maximum of three decimal places throughout:

1.
$$\int_1^7 x \ln x \, \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5-x^2} \, \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2+1} \, \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1+\sin x) \, \mathrm{d}x.$$

17.3.3 ANSWERS TO EXERCISES

1. 35.678   2. 0.882   3. 0.347   4. 1.469   5. 0.743