"JUST THE MATHS"

UNIT NUMBER

17.2

NUMERICAL MATHEMATICS 2
(Approximate integration (A))

by

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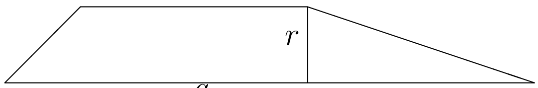
UNIT 17.2 - NUMERICAL MATHEMATICS 2

APPROXIMATE INTEGRATION (A)

17.2.1 THE TRAPEZOIDAL RULE

The rule which is explained below is based on the formula for the area of a trapezium. If the parallel sides of a trapezium are of length p and q while the perpendicular distance between them is r, then the area A is given by

$$A = \frac{r(p+q)}{2}.$$



Let us assume first that the curve whose equation is

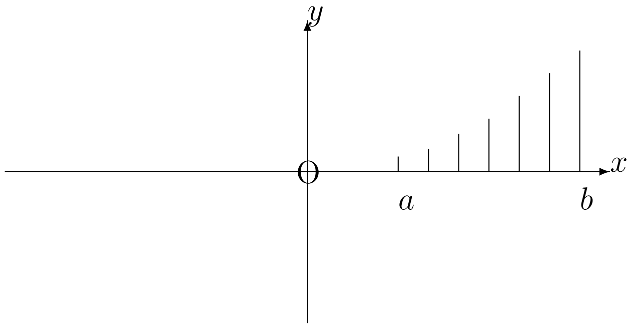
$$y = f(x)$$

lies wholly above the x-axis between $x = a$ and $x = b$. It has already been established, in Unit 13.1, that the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

can be regarded as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

However, suppose we divided this area into several narrow strips of equal width, h, by marking the values $x_1, x_2, x_3, ..., x_n$ along the x-axis (where $x_1 = a$ and $x_n = b$) and drawing in the corresponding lines of length $y_1, y_2, y_3, ..., y_n$ parallel to the y-axis.



Each narrow strip of width h may be considered approximately as a trapezium whose parallel sides are of lengths y_i and y_{i+1}, where $i = 1, 2, 3, ..., n - 1$.

Thus, the area under the curve, and hence the value of the definite integral, approximates to

$$\frac{h}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + ... + (y_{n-1} + y_n)].$$

That is,

$$\int_a^b f(x)\ \mathrm{d}x \simeq \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})];$$

or, what amounts to the same thing,

$$\int_a^b f(x)\ \mathrm{d}x = \frac{h}{2}[\text{First} + \text{Last} + 2 \times \ \text{The Rest}].$$

Note:
Care must be taken at the beginning to ascertain whether or not the curve $y = f(x)$ crosses the x-axis between $x = a$ and $x = b$. If it does, then allowance must be made for the fact that areas below the x-axis are negative and should be calculated separately from those above the x-axis.

EXAMPLE

Use the trapezoidal rule with five divisions of the x-axis in order to evaluate, approximately, the definite integral:

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

First we make up a table of values as follows:

x	0	0.2	0.4	0.6	0.8	1.0
e^{x^2}	1	1.041	1.174	1.433	1.896	2.718

Then, using $h = 0.2$, we have

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.2}{2}[1 + 2.718 + 2(1.041 + 1.174 + 1.433 + 1.896)] \simeq 1.481$$

17.2.2 EXERCISES

Use the trapezoidal rule with six divisions of the x-axis to determine an approximation for each of the following, working to three decimal places throughout:

1. $$\int_1^7 x \ln x\ \mathrm{d}x.$$

2. $$\int_{-2}^1 \frac{1}{5 - x^2}\ \mathrm{d}x.$$

3. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\ \mathrm{d}x.$$

4. $$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2 + 1}\ \mathrm{d}x.$$

5. $$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x)\ \mathrm{d}x.$$

17.2.3 ANSWERS TO EXERCISES

1. 35.836 2. 0.931 3. 0.348 4. 1.468 5. 0.737