16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

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UNIT 16.5 - LAPLACE TRANSFORMS 5

THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

\[
H(t) = \begin{cases} 
0 & \text{for } t < 0; \\
1 & \text{for } t > 0.
\end{cases}
\]

Note:

$H(t)$ is undefined when $t = 0$.

EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

\[
f(t) = \begin{cases} 
0 & \text{for } t < T; \\
1 & \text{for } t > T.
\end{cases}
\]
Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

**Note:**
The function $H(t-T)$ is of importance in constructing what are known as “pulse functions” (see later).

### 16.5.2 THE LAPLACE TRANSFORM OF $H(t - T)$

From the definition of a Laplace Transform,

$$L[H(t - T)] = \int_0^\infty e^{-st}H(t - T) \, dt$$

$$= \int_0^T e^{-st}0 \, dt + \int_T^\infty e^{-st}1 \, dt$$

$$= \left[ \frac{e^{-st}}{-s} \right]_T^\infty = \frac{e^{-sT}}{s}.$$

**Note:**
In the special case when $T = 0$, we have

$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.
16.5.3 PULSE FUNCTIONS

If \( a < b \), a “rectangular pulse”, \( P(t) \), of duration, \( b - a \), and magnitude, \( k \), is defined by the statements,

\[
P(t) = \begin{cases} 
  k & \text{for } a < t < b; \\
  0 & \text{for } t < a \text{ or } t > b.
\end{cases}
\]

We can show that, in terms of Heaviside functions, the above pulse may be represented by

\[P(t) \equiv k[H(t - a) - H(t - b)].\]

**Proof:**

(i) If \( t < a \), then \( H(t - a) = 0 \) and \( H(t - b) = 0 \). Hence, the above right-hand side = 0.

(ii) If \( t > b \), then \( H(t - a) = 1 \) and \( H(t - b) = 1 \). Hence, the above right-hand side = 0.

(iii) If \( a < t < b \), then \( H(t - a) = 1 \) and \( H(t - b) = 0 \). Hence, the above right-hand side = \( k \).

**EXAMPLE**

Determine the Laplace Transform of a pulse, \( P(t) \), of duration, \( b - a \), having magnitude, \( k \).

**Solution**

\[
L[P(t)] = k \left[ \frac{e^{-sa}}{s} - \frac{e^{-sb}}{s} \right] = k \cdot \frac{e^{-sa} - e^{-sb}}{s}.
\]
Notes:

(i) The “strength” of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, $k$. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t - a) - H(t - b)]f(t),$$

may be considered to “switch on” the function, $f(t)$, between $t = a$ and $t = b$ but “switch off” the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t - a)f(t),$$

may be considered to “switch on” the function, $f(t)$, when $t > a$ but “switch off” the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:

may be represented by the function

$$Q(t) \equiv k \{[H(t) - H(t - a)] + [H(t - 2a) - H(t - 3a)] + [H(t - 4a) - H(t - 5a)] + \ldots \ldots \}.$$
16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

\[ L[H(t - T)f(t - T)] = e^{-sT}L[f(t)]. \]

Proof:

Left-hand side =

\[ \int_0^\infty e^{-st}H(t - T)f(t - T) \, dt \]

\[ = \int_0^T 0 \, dt + \int_T^\infty e^{-st}f(t - T) \, dt \]

\[ = \int_T^\infty e^{-st}f(t - T) \, dt. \]

Making the substitution \( u = t - T \), we obtain

\[ \int_0^\infty e^{-s(u+T)}f(u) \, du \]

\[ = e^{-sT} \int_0^\infty e^{-su}f(u) \, du = e^{-sT}L[f(t)]. \]

EXAMPLES

1. Express, in terms of Heaviside functions, the function

\[ f(t) = \begin{cases} (t - 1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases} \]

and, hence, determine its Laplace Transform.

Solution

For values of \( t > 0 \), we may write

\[ f(t) = (t - 1)^2H(t - 1). \]

Therefore, using \( T = 1 \) in the second shifting theorem,

\[ L[f(t)] = e^{-s}L[t^2] = e^{-s} \cdot \frac{2}{s^3}. \]
2. Determine the inverse Laplace Transform of the expression

\[ \frac{e^{-7s}}{s^2 + 4s + 5}. \]

**Solution**
First, we find the inverse Laplace Transform of the expression,

\[ \frac{1}{s^2 + 4s + 5} \equiv \frac{1}{(s + 2)^2 + 1}. \]

From the first shifting theorem, this will be the function

\[ e^{-2t} \sin t, \quad t > 0. \]

From the second shifting theorem, the required function will be

\[ H(t - 7)e^{-2(t-7)} \sin(t - 7), \quad t > 0. \]

16.5.5 EXERCISES

1. (a) For values of \( t > 0 \), express, in terms of Heaviside functions, the function,

\[ f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases} \]

(b) Determine the Laplace Transform of the function, \( f(t) \), in part (a).

2. For values of \( t > 0 \), express, in terms of Heaviside functions, the function,

\[ f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases} \]

3. For values of \( t > 0 \), express the following functions in terms of Heaviside functions:

(a) \( f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases} \)

(b) \( f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases} \)
4. Use the second shifting theorem to determine the Laplace Transform of the function,

\[ f(t) \equiv t^3 H(t - 1). \]

**Hint:**
Write \( t^3 \equiv [(t - 1) + 1]^3 \).

5. Determine the inverse Laplace Transforms of the following:

(a) \[ \frac{e^{-2s}}{s^2}; \]

(b) \[ \frac{8e^{-3s}}{s^2 + 4}; \]

(c) \[ \frac{se^{-2s}}{s^2 + 3s + 2}; \]

(d) \[ \frac{e^{-3s}}{s^2 - 2s + 5}. \]

6. Solve the differential equation

\[ \frac{d^2 x}{dt^2} + 4x = H(t - 2), \]

given that \( x = 0 \) and \( \frac{dx}{dt} = 1 \) when \( t = 0 \).

16.5.6 ANSWERS TO EXERCISES

1. (a) \[ e^{-t}[H(t) - H(t - 3)]; \]

(b) \[ L[f(t)] = \frac{1 - e^{-3(s+1)}}{s + 1}. \]
2.  
\[ f(t) \equiv f_1(t)[H(t) - H(t - a)] + f_2(t)H(t - a). \]

3. (a)  
\[ f(t) \equiv t^2[H(t) - H(t - 2)] + 4tH(t - 2); \]
(b)  
\[ f(t) \equiv \sin t[H(t) - H(t - \pi)] + \sin 2t[H(t - \pi) - H(t - 2\pi)] + \sin 3t[H(t - 2\pi)]. \]

4.  
\[ L[f(t)] = \left[ \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s} \right] e^{-s}. \]

5. (a)  
\[ H(t - 2)(t - 2); \]
(b)  
\[ 4H(t - 3) \sin 2(t - 3); \]
(c)  
\[ H(t - 2)[2e^{-2(t-2)} - e^{-(t-2)}]; \]
(d)  
\[ \frac{1}{2}H(t - 3)e^{(t-3)} \sin 2(t - 3). \]

6.  
\[ x = \frac{1}{2} \sin 2t + \frac{1}{4}H(t - 2)[1 - \cos 2(t - 2)]. \]