"JUST THE MATHS"

UNIT NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

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UNIT 16.5 - LAPLACE TRANSFORMS 5

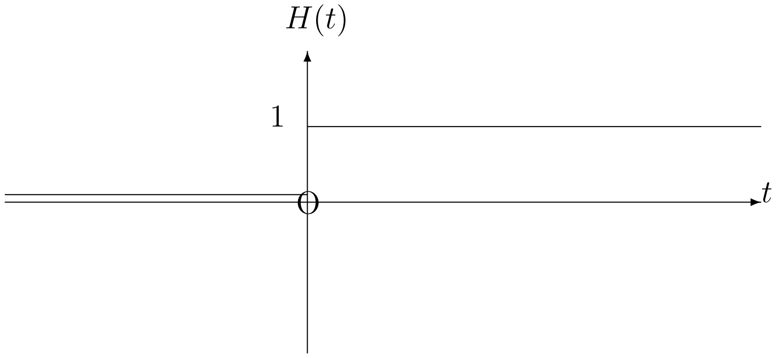
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note:
$H(t)$ is undefined when $t = 0$.

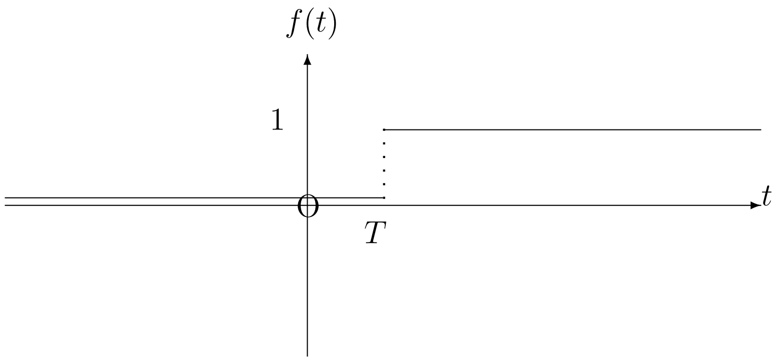


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function $H(t-T)$ is of importance in constructing what are known as **"pulse functions"** (see later).

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t-T)] = \int_0^\infty e^{-st} H(t-T) \ \mathrm{d}t$$

$$= \int_0^T e^{-st}.0 \ \mathrm{d}t + \int_T^\infty e^{-st}.1 \ \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$, we have

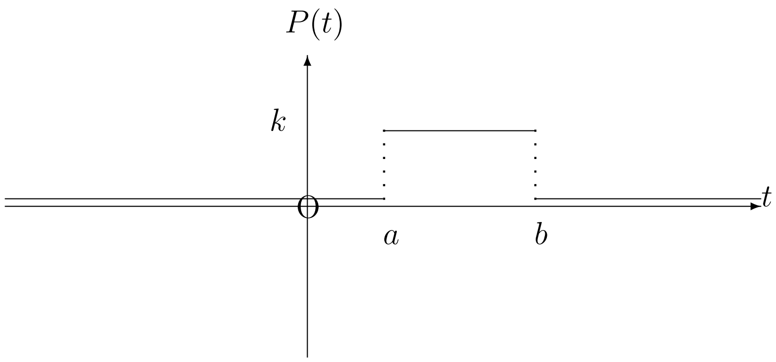
$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration, $b - a$, and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



We can show that, in terms of Heaviside functions, the above pulse may be represented by

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$. Hence, the above right-hand side = 0.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$. Hence, the above right-hand side = 0.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$. Hence, the above right-hand side = k.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration, $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right] = k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, k. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t-a) - H(t-b)]f(t),$$

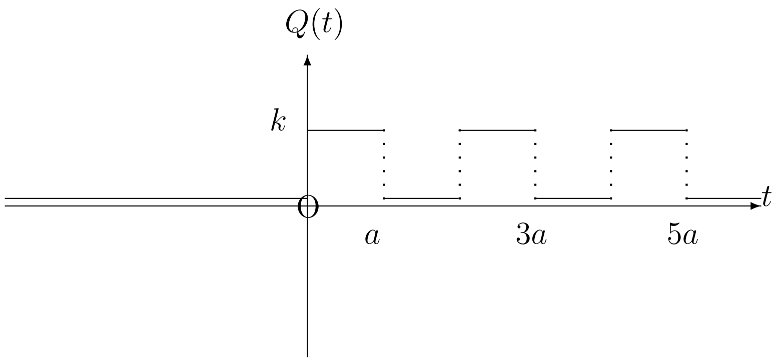
may be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t-a)f(t),$$

may be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:



may be represented by the function

$$Q(t) \equiv k\left\{[H(t) - H(t-a)] + [H(t-2a) - H(t-3a)] + [H(t-4a) - H(t-5a)] + ...\right\}.$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$

$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$

$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution $u = t - T$, we obtain

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$

$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we may write

$$f(t) = (t-1)^2 H(t-1).$$

Therefore, using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s}L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression

$$\frac{e^{-7s}}{s^2 + 4s + 5}.$$

Solution

First, we find the inverse Laplace Transform of the expression,

$$\frac{1}{s^2 + 4s + 5} \equiv \frac{1}{(s+2)^2 + 1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t} \sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)} \sin(t-7), \quad t > 0.$$

16.5.5 EXERCISES

1. (a) For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

   (b) Determine the Laplace Transform of the function, $f(t)$, in part (a).

2. For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases}$$

3. For values of $t > 0$, express the following functions in terms of Heaviside functions:

   (a)

$$f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases}$$

   (b)

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases}$$

4. Use the second shifting theorem to determine the Laplace Transform of the function,

$$f(t) \equiv t^3 H(t-1).$$

Hint:
Write $t^3 \equiv [(t-1)+1]^3$.

5. Determine the inverse Laplace Transforms of the following:

(a)

$$\frac{e^{-2s}}{s^2};$$

(b)

$$\frac{8e^{-3s}}{s^2+4};$$

(c)

$$\frac{se^{-2s}}{s^2+3s+2};$$

(d)

$$\frac{e^{-3s}}{s^2-2s+5}.$$

6. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = H(t-2),$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

16.5.6 ANSWERS TO EXERCISES

1. (a)

$$e^{-t}[H(t) - H(t-3)];$$

(b)

$$L[f(t)] = \frac{1 - e^{-3(s+1)}}{s+1}.$$

2.

$$f(t) \equiv f_1(t)[H(t) - H(t-a)] + f_2(t)H(t-a)].$$

3. (a)

$$f(t) \equiv t^2[H(t) - H(t-2)] + 4tH(t-2);$$

(b)

$$f(t) \equiv \sin t[H(t) - H(t-\pi)] + \sin 2t[H(t-\pi) - H(t-2\pi)] + \sin 3t[H(t-2\pi).$$

4.

$$L[f(t)] = \left[\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right] e^{-s}.$$

5. (a)

$$H(t-2)(t-2);$$

(b)

$$4H(t-3)\sin 2(t-3);$$

(c)

$$H(t-2)[2e^{-2(t-2)} - e^{-(t-2)};$$

(d)

$$\frac{1}{2}H(t-3)e^{(t-3)}\sin 2(t-3).$$

6.

$$x = \frac{1}{2}\sin 2t + \frac{1}{4}H(t-2)[1 - \cos 2(t-2)].$$