"JUST THE MATHS"

UNIT NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

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UNIT 16.8 - Z TRANSFORMS 1 - DEFINITION AND RULES

16.8.1 INTRODUCTION - Linear Difference Equations

Closely linked with the concept of a linear differential equation with constant coefficients is that of a **"linear difference equation with constant coefficients"**.

Two particular types of difference equation to be discussed in the present section may be defined as follows:

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1 are constants, n is a positive integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots.$

DEFINITION 2

A second-order linear difference equation with constant coefficients has the general form,

$$a_2 u_{n+2} + a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1, a_2 are constants, n is an integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots.$

Notes:

(i) We shall assume that the sequences under discussion are such that $u_n = 0$ whenever $n < 0$.

(ii) Difference equations are usually associated with given "boundary conditions", such as the value of u_0 for a first-order equation or the values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain **simple** difference equations may be solved by very elementary methods.

For example, suppose that we wish to solve the difference equation,

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$.

We may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n$$

and, by using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2, \quad u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \ . \ . \ .$$

In general, for this illustration, $u_n = n!$.

However, not all difference equations can be solved as easily as this and we shall now discuss the Z-Transform method of solving more advanced types.

16.8.2 STANDARD DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers, $\{u_n\} \equiv u_0, \ u_1, \ u_2, \ u_3, \ . \ . \ . \ .$, is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

$$\{u_n\} \equiv \{a^n\},$$

where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ .\ .\ .\ .\ = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a},$$

by properties of infinite geometric series.

Thus,

$$Z\{a^n\} = \frac{z}{z-a}.$$

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} r z^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ .\ .\ .\ .\ ,$$

which may be rearranged as

$$Z\{n\} = \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^2}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^3}}{1 - \frac{1}{z}} + \ .\ .\ .\ . = \frac{1}{1 - \frac{1}{z}} \left[\frac{\frac{1}{z}}{1 - \frac{1}{z}}\right],$$

by properties of infinite geometric series.

Thus,

$$Z\{n\} = \frac{z}{(1-z)^2} = \frac{z}{(z-1)^2}.$$

Note:
Other Z-Transforms may be obtained, in the same way as in the above examples, from the definition.

We list, here, for reference, a short table of standard Z-Transforms, including those already proven:

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region of Existence
$\{1\}$	$\frac{z}{z-1}$	$\|z\| > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\left\{e^{-nT}\right\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\|z\| > e^{-T}$
$\sin nT$ (T constant)	$\frac{z \sin T}{z^2-2z\cos T+1}$	$\|z\| > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\|z\| > 1$
1 for $n = 0$ 0 for $n > 0$ (Unit pulse sequence)	1	All z
0 for $n = 0$ $\{a^{n-1}\}$ for $n > 0$	$\frac{1}{z-a}$	$\|z\| > \|a\|$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} (Au_r + Bv_r)z^{-r} \equiv A\sum_{r=0}^{\infty} u_r z^{-r} + B\sum_{r=0}^{\infty} v_r z^{-r},$$

which, in turn, is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1, 2, 3, 4, \ldots.$, are the terms $u_0, u_1, u_3, u_4, \ldots$ of the original sequence.

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1} z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ldots.,$$

since it is assumed that $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ \ldots .\right],$$

which is equivalent to the right-hand side.

Note:
A more general form of the first shifting theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to $n = 0,\ 1,\ 2,\ \ldots . , k-1$, are taken as zero and whose subsequent terms, corresponding to $n = k,\ k+1,\ k+2,\ \ldots .$ are the terms $u_0,\ u_1,\ u_2,\ \ldots .$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
In this illustration, the sequence, $\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ \ldots$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5}\ \ldots .,$$

which gives

$$Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series.

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

This may be rearranged as

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

which, in turn, is equivalent to the right-hand side.

Note:
This **"recursive relationship"** may be applied repeatedly. For example, we may deduce that

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

16.8.4 EXERCISES

1. Determine, from first principles, the Z-Transforms of the following sequences, $\{u_n\}$:

   (a)

   $$\{u_n\} \equiv \{e^{-n}\};$$

   (b)

   $$\{u_n\} \equiv \{\cos \pi n\}.$$

2. Determine the Z-Transform of the following sequences:

   (a)
$$\{u_n\} \equiv \{7.(3)^n - 4.(-1)^n\};$$

   (b)
$$\{u_n\} \equiv \left\{6n + 2e^{-5n}\right\};$$

   (c)
$$\{u_n\} \equiv \{13 + \sin 2n - \cos 2n\}.$$

3. Determine the Z-Transform of $\{u_{n-1}\}$ and $\{u_{n-2}\}$ for the sequences in question 1.

4. Determine the Z-Transform of $\{u_{n+1}\}$ and $\{u_{n+2}\}$ for the sequences in question 1.

16.8.5 ANSWERS TO EXERCISES

1. (a)
$$\frac{ez}{ez-1};$$

   (b)
$$\frac{z}{z+1}.$$

2. (a)
$$\frac{7z}{z-3} - \frac{4z}{z+1};$$

   (b)
$$\frac{6z}{(z-1)^2} + \frac{2z}{z-e^{-5}};$$

   (c)
$$\frac{13z}{z-1} + \frac{z(\sin 2 + \cos 2 - z)}{z^2 - 2z\cos 2 + 1}.$$

3. (a)

$$Z\{u_{n-1}\} \equiv \frac{e}{ez-1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{e}{z(ez-1)}\ (n>1);$$

(b)

$$Z\{u_{n-1}\} \equiv \frac{1}{z+1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{1}{z(z+1)}\ (n>1).$$

Note:
$u_{-2} = 0$ and $u_{-1} = 0$.

4. (a)

$$Z\{u_{n+1}\} \equiv \frac{z}{ez-1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{e(ez-1)};$$

(b)

$$Z\{u_{n+1}\} \equiv -\frac{z}{z+1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{z+1}.$$