"JUST THE MATHS"

UNIT NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

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One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) Let us consider that our main problem is to solve a second order linear differential equation with constant coefficients, the general form of which is

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+b\frac{\mathrm{d}x}{\mathrm{d}t}+cx=f(t).$$

(ii) Assuming that the solution of an equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=f(t),$$

has already been included in previous knowledge, we examine a typical worked example as follows:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t}+3x=e^{2t},$$

given that $x=0$ when $t=0$.

Solution

A method called the **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation.

The integrating factor in the current example is e^{3t} since the coefficient of x is 3.

We obtain, therefore,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t}+3x\right]=e^{5t}.$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right]=e^{5t}.$$

On integrating both sides with respect to t,

$$xe^{3t}=\frac{e^{5t}}{5}+C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$ and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) As a lead up to what follows, we shall now examine a different way of setting out the above working in which we do not leave the substitution of the boundary condition until the very end.

We multiply both sides of the differential equation by e^{3t} as before, but we then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] \mathrm{d}t = \int_0^t e^{5t} \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5}.$$

since $x = 0$ when $t = 0$.

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before.

(iv) Now let us consider whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Supposing that there might be an integrating factor for this equation, we shall take it to be e^{st} where s, at present, is unknown, but assumed to be positive.

Multiplying throughout by e^{st} and integrating from 0 to t, as in the previous example,

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right]\ \mathrm{d}t = \int_0^t e^{(s+9)t}\ \mathrm{d}t = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Now, using integration by parts, with the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = e^{st}x - s\int_0^t e^{st}x\ \mathrm{d}t$$

and

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\ \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x\ \mathrm{d}t.$$

On substituting these results into the differential equation, we may collect together (on the left hand side) terms which involve $\int_0^t e^{st}x\ \mathrm{d}t$ and e^{st} as follows:

$$(s^2 + 10s + 21)\int_0^t e^{st}x\ \mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \frac{e^{(s+9)t}}{s+9} - \frac{1}{s+9}.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation; that is, $(s^2 - 10s + 21)$.

(b) Using e^{-st} with $s > 0$, if we had integrated from 0 to ∞ instead of 0 to t, the second term on the left hand side above would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞ to obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \text{ d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Of course, this works only if $s > 9$, but we can easily assume that it is so. Hence,

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{(s-9)(s^2-10s+21)} = \frac{1}{(s-9)(s-3)(s-7)}.$$

Applying the principles of partial fractions, we obtain

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) But, finally, it can be shown by an independent method of solution that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

and we may conclude that the solution of the differential equation is closely linked to the integral

$$\int_0^\infty e^{-st}x \text{ d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.