"JUST THE MATHS"

UNIT NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

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UNIT 16.4 - LAPLACE TRANSFORMS 4
SIMULTANEOUS DIFFERENTIAL EQUATIONS

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In this Unit, we shall consider a pair of differential equations involving an independent variable, t, such as a time variable, and two dependent variables, x and y, such as electric currents or linear displacements.

The general format is as follows:

$$\begin{aligned} a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t), \\ a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t). \end{aligned}$$

To solve these equations simultaneously, we take the Laplace Transform of each equation obtaining two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t, \end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations,

$$sY(s) - 2 + 2X(s) = \frac{1}{s-1},$$

$$sX(s) - 1 - 2Y(s) = \frac{1}{s} + \frac{1}{s^2}.$$

That is,

$$\begin{aligned} 2X(s) + sY(s) &= \frac{1}{s-1} + 2, & (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. & (2) \end{aligned}$$

Using $(1) \times 2 + (2) \times s$, we obtain

$$(4 + s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Applying the methods of partial fractions, this gives

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y, and this is often necessary; but, since

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t$$

in the current example,

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

16.4.2 EXERCISES

Use Laplace Transforms to solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2y &= e^{-t},\\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

2.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} - y &= \sin t,\\ \frac{\mathrm{d}y}{\mathrm{d}t} + x &= \cos t,\end{aligned}$$

given that $x = 3$ and $y = 4$ when $t = 0$.

3.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 3y &= 1,\\ \frac{\mathrm{d}y}{\mathrm{d}t} - x + 2y &= e^{-2t},\end{aligned}$$

given that $x = 0$ and $y = 0$ when $t = 0$.

4.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 8x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

5.

$$\begin{aligned} 10\frac{\mathrm{d}x}{\mathrm{d}t} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 6x + 5y &= 0, \\ 2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} + 2x + y &= 2e^{-t}, \end{aligned}$$

given that $x = 2$ and $y = -1$ when $t = 0$.

6.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y &= 0, \\ 5\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 6x &= 0, \end{aligned}$$

given that $x = 3$ and $y = 0$ when $t = 0$.

7.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2z, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= 2x, \end{aligned}$$

given that $x = 1$, $y = 0$ and $z = -1$ when $t = 0$.

16.4.3 ANSWERS TO EXERCISES

1.

$$x = (2t+1)e^{-t} \quad \text{and} \quad y = te^{-t}.$$

2.

$$x = (t+4)\sin t + 3\cos t \quad \text{and} \quad y = (t+4)\cos t - 3\sin t.$$

3.

$$x = 2 - e^{-2t}[1 + \sqrt{3}\sinh t\sqrt{3} + \cosh t\sqrt{3}]$$

and

$$y = 1 - e^{-2t}\left[\cosh t\sqrt{3} + \frac{1}{\sqrt{3}}\sinh t\sqrt{3}\right].$$

4.

$$x = \sinh 4t \quad \text{and} \quad y = 2\cosh 4t.$$

5.

$$x = 4\cos t - 2e^{-t} \quad \text{and} \quad y = e^{-t} - 2\cos t.$$

6.

$$x = 2e^{-t} + e^{-2t} \quad \text{amd} \quad y = e^{-t} - e^{-2t}.$$

7.

$$x = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} + \cos t\sqrt{3}\right],$$

$$y = \frac{-2}{\sqrt{3}}e^{-t}\sin t\sqrt{3}$$

and

$$z = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} - \cos t\sqrt{3}\right].$$