"JUST THE MATHS"

UNIT NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

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UNIT 16.3 - LAPLACE TRANSFORMS 3 - DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

In the work which follows, the problems considered will usually take the form of a linear differential equation of the second order with constant coefficients.

That is,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

However, the method will apply equally well to the corresponding first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t).$$

The technique will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square to obtain

$$X(s) \equiv \frac{3s + 12}{(s + 2)^2 + 9} \equiv \frac{3(s + 2) + 6}{(s + 2)^2 + 9} \equiv 3.\frac{s + 2}{(s + 2)^2 + 9} + 2.\frac{3}{(s + 2)^2 + 9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2+1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2+1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

Thus,

$$X(s) \equiv \frac{50}{(s^2+6s+9)(s^2+1)} + \frac{s+10}{s^2+6s+9}$$

or

$$X(s) \equiv \frac{50}{(s+3)^2(s^2+1)} + \frac{s+10}{(s+3)^2}.$$

Using the principles of partial fractions in the first term on the right,

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Hence,

$$50 \equiv A(s^2+1) + B(s+3)(s^2+1) + (Cs+D)(s+3)^2.$$

Substituting $s = -3$,

$$50 = 10A \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides (we shall not need the s^2 coefficients in this example),

$$0 = B + 9C + 6D. \qquad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \qquad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0, \qquad (4)$$

and we already have

$$3B + 9D = 45. \qquad (3)$$

These last two solve easily to give $B = 3$ and $D = 4$ so that $C = -3$.

We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition to this, we also have

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1}.$$

This gives

$$(s^2 + 4s - 3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2+4s-3) + (Bs+C)(s-1).$$

Substituting $s = 1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may either be given no boundary conditions at all; or else the boundary conditions given do not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we simply let $x(0) = A$ and $x'(0) = B$ to obtain a solution in terms of A and B called the **"general solution"**.

If any non-standard boundary conditions are provided, we then substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0;$$

but, since A and B are **arbitrary** constants, this may be written in the simpler form

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0,$$

in which $\frac{B}{2}$ has been rewritten as B.

To apply the boundary conditions, we require also the formula for $x'(t)$, namely

$$x'(t) = -\,2A\sin 2t + 2B\cos 2t.$$

Hence, $-\,3 = -\,A$ and $10 = -\,2B$ giving $A = 3$ and $B =$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

16.3.3 EXERCISES

1. Solve the following differential equations subject to the conditions given:

   (a)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 0,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (b)
   $$4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$
   given that $x(0) = 4$ and $x'(0) = 1$;

   (c)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - 8x = 2t,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (d)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 2e^{2t},$$
   given that $x(0) = 1$ and $x'(0) = 10.5$;

   (e)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 3\cos^2 t,$$
   given that $x(0) = 1$ and $x'(0) = 2$.
   Hint: $\cos 2t \equiv 2\cos^2 t - 1$.

2. Determine the particular solution of the differential equation
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} = e^t(t-3)$$
   in the case when $x(0) = 2$ and $x(3) = -1$.
   Hint:
   Since $x(0)$ is given, just let $x'(0) = B$ to obtain a solution in terms of B; then substitute the second boundary condition at the end.

16.3.4 ANSWERS TO EXERCISES

1. (a)

$$X(s) = \frac{3s-5}{s^2-2s+5},$$

giving

$$x(t) = e^t(3\cos 2t - \sin 2t) \quad t > 0;$$

(b)

$$X(s) = \frac{4}{s+\frac{1}{2}} + \frac{3}{(s+\frac{1}{2})^2},$$

giving

$$x(t) = 4e^{-\frac{1}{2}t} + 3te^{-\frac{1}{2}t} = e^{-\frac{1}{2}t}[4+3t] \quad t > 0;$$

(c)

$$X(s) = \frac{27}{12}.\frac{1}{s-2} + \frac{39}{48}.\frac{1}{s+4} - \frac{1}{4}.\frac{1}{s^2} - \frac{1}{16}.\frac{1}{s},$$

giving

$$x(t) = \frac{27}{12}e^{2t} + \frac{39}{48}e^{-4t} - \frac{1}{4}t - \frac{1}{16} \quad t > 0;$$

(d)

$$X(s) = \frac{\frac{1}{2}}{(s-2)^2} + \frac{3}{s-2} - \frac{2}{s+2},$$

giving

$$x(t) = \frac{1}{2}te^{2t} + 3e^{2t} - 2e^{-2t} \quad t > 0;$$

(e)

$$X(s) = \frac{3}{2}.\frac{s}{(s^2+4)^2} + \frac{3}{8}.\frac{1}{s} + \frac{5}{8}.\frac{s}{s^2+4} + \frac{2}{s^2+4},$$

giving

$$x(t) = \frac{3}{8}t\sin 2t + \frac{3}{8} + \frac{5}{8}\cos 2t + \sin 2t \quad t > 0.$$

2.

$$x(t) = 3e^t - te^t - 1 \quad t > 0.$$