"JUST THE MATHS"

UNIT NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

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UNIT 16.2 - LAPLACE TRANSFORMS 2
INVERSE LAPLACE TRANSFORMS

In order to solve differential equations, we now examine how to determine a function of the variable, t, whose Laplace Transform is already known.

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORMS

A function of t, whose Laplace Transform is the given expression, $F(s)$, is called the **"Inverse Laplace Transform"** of $f(t)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:
(i) Since two functions which coincide for $t > 0$ will have the same Laplace Transform, we can determine the Inverse Laplace Transform of $F(s)$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear** since

$$L^{-1}\left[AF(s) + BG(s)\right]$$

is a function of t whose Laplace Transform is

$$AF(s) + BG(s);$$

and, by the linearity of Laplace Transforms, discussed in Unit 16.1, such a function is

$$AL^{-1}[F(s)] + BL^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

The type of differential equation to be encountered in simple practical problems usually lead to Laplace Transforms which are **"rational functions of s"**. We shall restrict the discussion to such cases, as illustrated in the following examples, where the table of standard Laplace Transforms is used whenever possible. The partial fractions are discussed in detail, but other, shorter, methods may be used if known (for example, the "Cover-up Rule" and "Keily's Method"; see Unit 1.9)

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s+3 \equiv A(s+3) + Bs$$

Note:

Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions. Otherwise, entire partial fractions exercises would have to be carried out by equating coefficients of appropriate powers of s on both sides.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B;$$

so that

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2+9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2+5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}}\sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2 + 4) + (Bs + C)(s + 1).$$

Substituting $s = -1$, we obtain

$$5 = 5A \text{ which implies that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24} t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t} \sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**, giving

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

In order to use the First Shifting Theorem, we must try to include $s+2$ in the numerator; so we write

$$F(s) = \frac{(s+2) - 2}{(s+2)^2 + 9} = \frac{s+2}{(s+2)^2 + 3^2} - \frac{2}{3} \cdot \frac{3}{(s+2)^2 + 3^2}.$$

Hence,

$$f(t) = e^{-2t} \cos 3t - \frac{2}{3} e^{-2t} \sin 3t = \frac{1}{3} e^{-2t} \left[3 \cos 3t - 2 \sin 3t\right] \quad t > 0.$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2 + 4s + 8)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{8(s+1)}{s(s^2 + 4s + 8)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2 + 4s + 8) + (Bs + C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s + 4}{s^2 + 4s + 8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square to give

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

which, on rearrangement, becomes

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

Thus, from the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$, and partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s+10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s = -2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s = 6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

However, if we did not factorise the denominator, a different form of solution could be obtained as follows:

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\text{cosh}4t + 3\text{sinh}4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform of this function could certainly be obtained by using partial fractions, but we note here how it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\ \mathrm{d}T = \int_0^t e^{(3T-2t)}\ \mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

16.2.3 EXERCISES

Determine the Inverse Laplace Transforms of the following rational functions of s:

1. (a)

$$\frac{1}{(s-1)^2};$$

(b)

$$\frac{1}{(s+1)^2+4};$$

(c)

$$\frac{s+2}{(s+2)^2+9};$$

(d)

$$\frac{s-2}{(s-3)^3};$$

(e)

$$\frac{1}{(s^2+4)^2};$$

(f)

$$\frac{s+1}{s^2+2s+5};$$

(g)

$$\frac{s-3}{s^2-4s+5};$$

(h)

$$\frac{s-3}{(s-1)^2(s-2)};$$

(i)

$$\frac{5}{(s+1)(s^2-2s+2)};$$

(j)

$$\frac{2s-9}{(s-3)(s+2)};$$

(k)

$$\frac{3}{s(s^2+9)};$$

(l)

$$\frac{2s-1}{(s-1)(s^2+2s+2)}.$$

2. Use the Convolution Theorem to obtain the Inverse Laplace Transform of

$$\frac{s}{(s^2+1)^2}.$$

16.2.4 ANSWERS TO EXERCISES

1. (a)

$$te^t \quad t > 0;$$

(b)

$$\frac{1}{2}e^{-t}\sin 2t \quad t > 0;$$

(c)

$$e^{-2t}\cos 3t \quad t > 0;$$

(d)

$$e^{3t}\left[t + \frac{1}{2}t^2\right] \quad t > 0;$$

(e)

$$\frac{1}{16}[\sin 2t - 2t\cos 2t] \quad t > 0;$$

(f)

$$e^{-t}\cos 2t \quad t > 0;$$

(g)

$$e^{2t}[\cos t - \sin t] \quad t > 0;$$

(h)

$$2te^{t} + e^{t} - e^{2t} \quad t > 0;$$

(i)

$$e^{-t} + e^{t}[2\sin t - \cos t] \quad t > 0;$$

(j)

$$\frac{1}{5}[13e^{-2t} - 3e^{3t}] \quad t > 0;$$

(k)

$$\frac{1}{3}[1 - \cos 3t] \quad t > 0;$$

(l)

$$\frac{1}{5}[e^{t} - e^{-t}\cos t + 8e^{-t}\sin t] \quad t > 0.$$

2.

$$\frac{1}{2}t\sin t \quad t > 0.$$